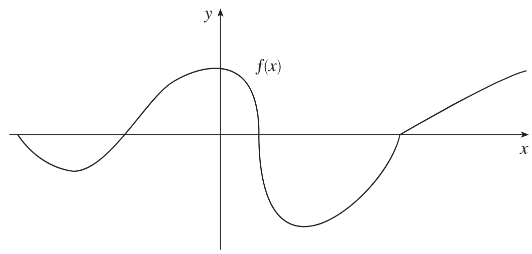
Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the function $f(x) = \begin{cases} -x & \text{if } x \le 1 \\ x - 2 & \text{if } x > 1 \end{cases}$ Then $f(0) = f(2) = 0$. Your friend claims that by the Mean Value Theorem, $f'(c)$ should be zero for some c with $0 < c < 2$.

(a) Find such a value of c or show why it does not exist.

(b) Does your answer to part (a) contradict the Mean Value Theorem? Give reasons for your answer.

2. Identify the inflection points for $f(x)$ on the following picture:



At each inflection point, give possible values for $f''(x)$, or explain why it does not exist.

3. Find the antiderivative F for $f(x) = |\sin x|$, $-\pi \le x \le \pi$ satisfying $F(0) = 1$.

4. For each of the following functions, identify all vertical and horizontal asymptotes, as well as slant asymptotes when applicable:

(a) $f(x) = \dfrac{x^2 + 3}{(x+3)^2 (x-1)}$

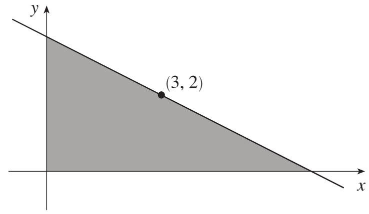
(b) $g(x) = \dfrac{3 - 4x^2}{x^2 - x - 6}$

(c) $h(x) = \dfrac{x^3 + 2x^2 - x - 2}{x^2 - 4}$

(d) $k(x) = \dfrac{x + 1 - \sin x}{x - 1}$

(e) $m(x) = \sqrt{\dfrac{x^4 + 1}{x^2 - 2}}$

5. A triangle is formed by a line through (3, 2) and the two coordinate axes.



(a) Is there a minimal area for such a triangle? If so, what is it? If not, why not?

(b) Is there a maximal area for such a triangle? If so, what is it? If not, why not?

6. Let f be a function which is continuous on the interval $[0, 5]$, twice differentiable on $(0, 5)$, and has $f''(x) < 0$ for all $x \in (0, 5)$. Which of the following are (i) always true, (ii) never true, or (iii) sometimes true? Justify your answers.

(a) For some c with $0 < c < 5$, $f'(c) = \frac{1}{5}\left[f(5) - f(0)\right]$.

(b) f has an inflection point somewhere in $(0, 5)$.

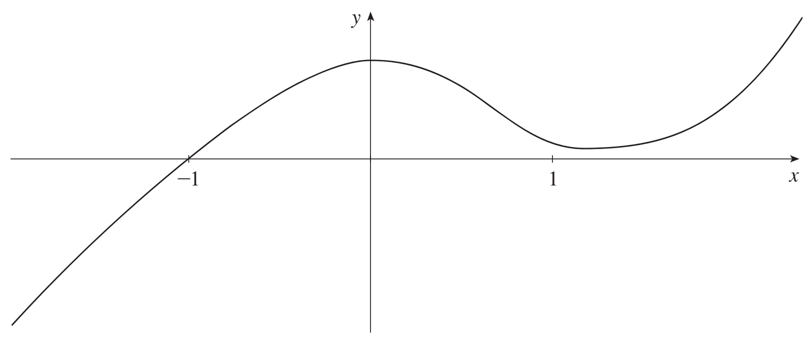
(c) f achieves its maximum at $x = 5$.

(d) f' achieves its minimum at $x = 5$.

7. Let $f(x)$ be the function graphed below, which has a root at $x = -1$.

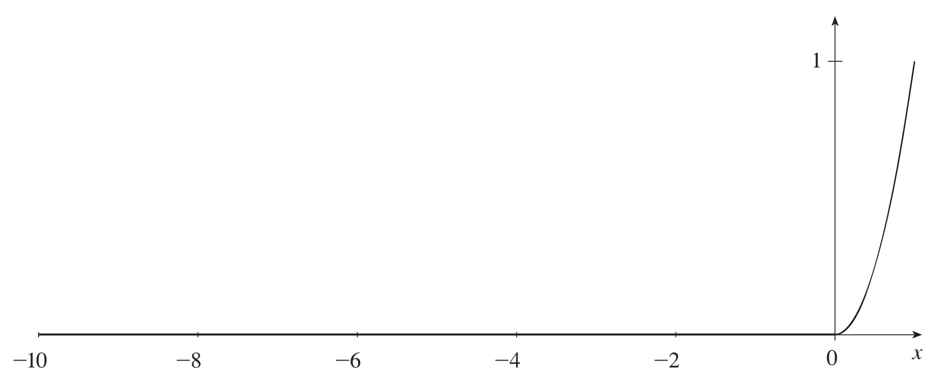
(a) Why does Newton's method work if we make the initial guess of $x_0 = -0.5$?

(b) Why might Newton's method fail to find this root if we make the initial guess of $x_0 = 0.5$?



8. (a) Let f be a function such that $f(1) = 5$ and $f(3) = 5$. What additional conditions on f will guarantee that there is a point between $x = 3$ and $x = 5$ where f has a horizontal tangent line?

(b) Let $f(x) = \begin{cases} 0 & \text{if } -10 \le x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \end{cases}$



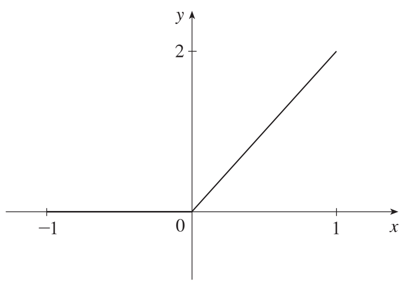
(i) Show that this function is differentiable at $x = 0$.

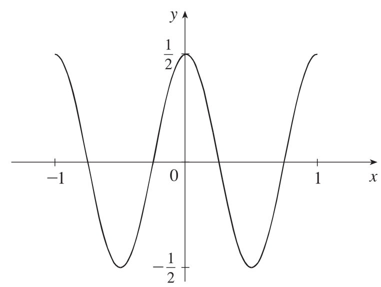
(ii) Find a value of c with $-10 < c < 1$ for which the conclusion of the Mean Value Theorem is true.

9. Let g be a differentiable function defined everywhere with the following values

$$g(-1) = -1 \qquad g(0) = 0 \qquad g(1) = 0$$

The graphs of two functions are shown below. For each graph explain why it can or cannot be the graph of g', the derivative of g.





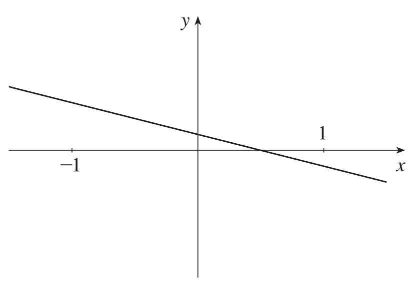
10. Let $f(x) = x^3 + ax^2 + bx + c$ where a, b, and c are constants with $a \geq 0$ and $b > 0$.

(a) Over what intervals is f concave up? Concave down?

(b) Show that f must have exactly one inflection point.

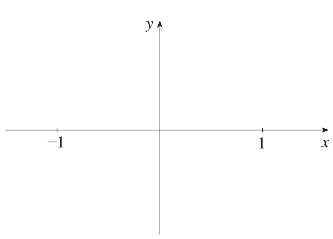
(c) Given that $(0, -2)$ is the inflection point of f, compute a and c and then show that f has no critical point.

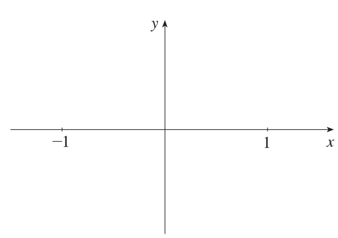
11. Let $f(x)$ be a function whose *second* derivative f'' is sketched below.

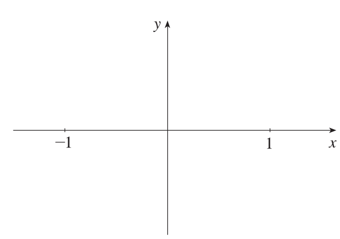


Sketch possible graphs of f such that

(a) f is increasing on $[-1, 1]$. (b) f has a local minimum at $x = 0$. (c) f is decreasing on $[-1, 1]$.







12. Suppose that your foot length L in inches is related to your height h in inches by $L = \frac{4}{3}\sqrt{h}$. In one (non-leap) year, you have a growth spurt in which you grow from 64 inches to 69 inches. For simplicity of modeling, assume that your height changes at a constant rate throughout the year.

(a) Find an expression for $\dfrac{dL}{dt}$.

(b) What was the fastest rate of growth that your foot experienced during this time? When did it occur?

13. Let $f(x) = \dfrac{1}{1+x^4} + A$, where A is a constant, and let $F(x)$ be an antiderivative of f.

(a) Suppose that $A = 2$. Show that F has no critical point.

(b) Find a value of A for which F has exactly one critical point.

(c) Find a value of A for which F has exactly two critical points.

14. We are going to use Newton's Method to approximate $\sqrt[3]{a}$ for $a > 0$.

(a) Let $f(x) = x^3 - a$. Use Newton's Method to write a formula for x_{n+1} [the $(n+1)$st "guess"] in terms of x_n.

(b) Show that your answer to part (b) can be written as

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right)$$

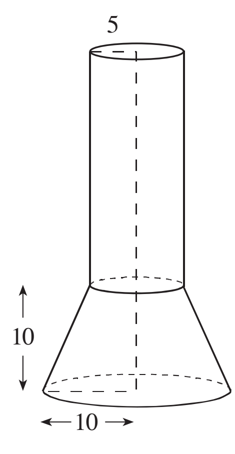
(c) Using $x_0 = 3$ as your initial guess, use Newton's Method to find three better approximations to $\sqrt[3]{29}$.

(d) Compute the error between the third approximation and the value your calculator gives for $\sqrt[3]{29}$. You might be interested to know that the fourth approximation is given by the fraction

$$\frac{10362169620245014291032874414215614785182224248780023228301}{3372754246441305714003762879114472623803566295847119940489}$$

which is correct to 26 decimal places.

*** 15.** A flask has the shape sketched below. The bottom part of the flask is in the shape of a truncated cone with bottom radius 10 cm, top radius 5 cm, and height 10 cm. Sitting on top of the truncated cone is a tall cylinder whose radius is 5 cm. Liquid is being poured into the flask at a rate of 2 cm^3/s. Note that the volume of a cone whose bottom radius is r and whose vertical height is h is given by $V = \frac{1}{3}\pi r^2 h$.



(a) When the liquid is 15 cm high in the flask, how fast is the height of the liquid increasing?

(b) If the flask is initially empty, how long does it take the liquid to reach a height of 10 cm?

(c) How fast is the height of the liquid rising when the liquid is 5 cm deep?

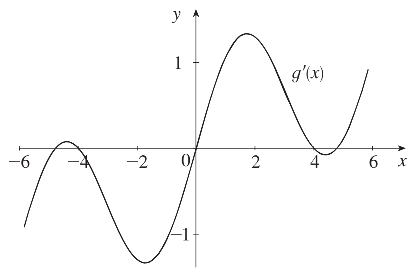
*** 16.** Let $f(x)$ be a function that is continuous on the interval $[-5, 15]$, has continuous first and second derivatives on $(-5, 15)$, and has values $f(0) = 3$, $f(5) = -2$, and $f(10) = 8$.

(a) Sketch a possible graph of f.

(b) Prove that there is a point a in $(0, 10)$ such that $f'(a) < 0$ and that there is a point b in $(0, 10)$ such that $f'(b) > 0$. Show your work.

(c) Using part (b), prove there is a point c in $(0, 10)$ such that $f'(c) = 0$. Show your work.

17. Consider the graph of g', the derivative of g.



(a) Find and classify the local extrema of g.

(b) Where is g concave up?

(c) Which is larger, $g(2)$ or $g(4)$, or is there not enough information to tell?

18. Consider the function f, whose formula and derivatives are given below:

$$f(x) = \frac{2x^2 - 5x + 5}{x - 2} \qquad f'(x) = \frac{2x^2 - 8x + 5}{(x-2)^2} \qquad f''(x) = \frac{6}{(x-2)^3}$$

(a) Find and describe all of the vertical, horizontal and slant asymptotes of this function, if any.

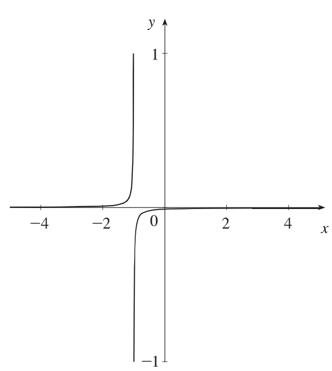
(b) Find all of the roots of this functions, if any.

(c) Find all of the local extrema of this function, if any.

(d) Find all of the inflection points of this function, if any.

(e) Sketch this function, including all of the features above in your sketch

19. A student graphs the function $\dfrac{1}{x^2 - 99x - 100}$ on her calculator and sees the following:



Is this graph sufficient to get an idea of what the function looks like? Why or why not?

20. Let f be a continuous, differentiable function with $f(0) = 0$ and $-1 \leq f'(x) \leq 5$ for $-3 \leq x \leq 3$.

(a) Can $f(3)$ ever be negative? Give a reason for your answer.

(b) Can $f(-3)$ ever be negative? Give a reason for your answer.

(c) How large can $f(3)$ be? How large can $f(-3)$ be?

(d) Must f have a critical point between -3 and 3? Either show your reasoning or give a counterexample.

21. Let $g(x) = \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}$

(a) Compute $g'(x)$.

(b) Rewrite your answer to $g'(x)$ in terms of $\sin x$ and $\cos x$.

(c) Using part b, rewrite the original function in terms of $\sin x$ and $\cos x$.

22. Give examples of functions $f(x)$ and $g(x)$ with $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$ and

(a) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

(b) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 6$

(c) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

(d) Is it possible to have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = -1$? Either give an example or explain why it is not possible.