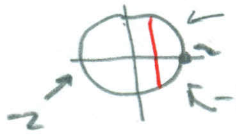
Name Key

AB Calculus

Cross – Sectional Volume

1. Find the volume of a solid whose base is the region inside the circle $x^2 + y^2 = 4$ and the cross sections taken are perpendicular to the x-axis are equilateral triangles.



$\sqrt{4-x^2} = y$

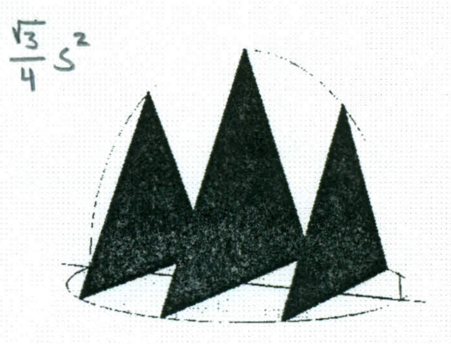
$-\sqrt{4-x^2} = y$

-2

s s

$A(x) = \frac{\sqrt{3}}{4} s^2$

$s = 2\sqrt{4-x^2}$

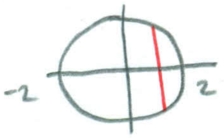


$$V(x) = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} \frac{\sqrt{3}}{4}\, 4(4-x^2)\,dx$$

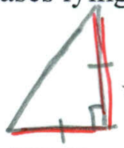
$$\sqrt{3}\int_{-2}^{2} x^2-4\,dx \qquad \sqrt{3}\left[4x - \frac{1}{3}x^3 \Big|_{-2}^{2}\right]$$

$$\sqrt{3}\left[\left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right)\right] = \sqrt{3}\cdot\left(16-\frac{16}{3}\right) = \frac{\sqrt{3}\cdot 32}{3} \quad \boxed{\frac{32\sqrt{3}}{3}}$$

2. Find the volume of a solid whose base is the region inside the circle $x^2 + y^2 = 4$ and the cross sections taken are perpendicular to the x-axis are isosceles right triangles with bases lying along the base of the circle.



-2 2



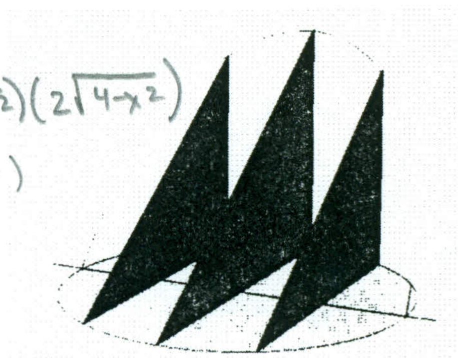
$2\sqrt{4-x^2}$

$2\sqrt{4-x^2}$

$$A(x) = \frac{1}{2}bh$$

$$= \frac{1}{2}\left(2\sqrt{4-x^2}\right)\left(2\sqrt{4-x^2}\right)$$

$$= 2(4-x^2)$$

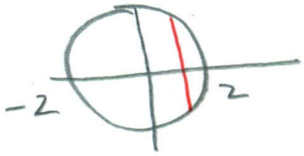


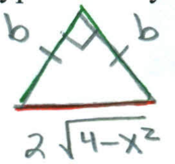
$$2\int_{0}^{2} 2(4-x^2)\,dx$$

symmetry

$$4\left[\left(4x - \frac{1}{3}x^3\right)\Big|_{0}^{2}\right] = 4\left(8-\frac{8}{3}\right) = 4\cdot\frac{16}{3} = \boxed{\frac{64}{3}}$$

3. Find the volume of a solid whose base is the region inside the circle $x^2 + y^2 = 4$ and the cross sections taken are perpendicular to the x-axis are isosceles right triangles whose hypotenuse lying along the base of the circle.





$$A(x) = \tfrac{1}{2} b h$$

$$= \tfrac{1}{2}\left(\frac{4(4-x^2)}{2}\right)$$

$$= 4 - x^2$$

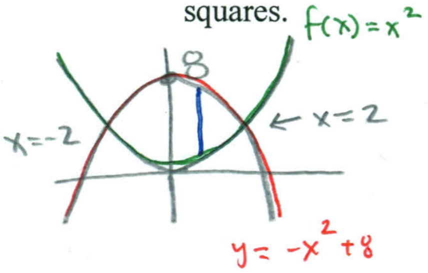
$$\frac{b}{1} = \frac{2\sqrt{4-x^2}}{\sqrt{2}}$$

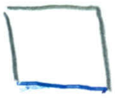
$$2\int_0^2 (4-x^2)\,dx \qquad 2\left[4x - \tfrac{1}{3}x^3\right]\Big|_0^2$$

$$2\left(8 - \tfrac{8}{3}\right) = \boxed{\frac{32}{3}}$$

4. The base of a solid is the region bounded by the graphs of $f(x) = x^2$ and $g(x) = 8 - x^2$

a. Find the volume of the solid if all cross sections perpendicular to the x-axis are squares.





$-x^2 + 8 - x^2$

$-2x^2 + 8$

$$A(x) = s^2$$

$$= (-2x^2+8)^2$$

$$= 4x^4 - 32x^2 + 64$$

$$2\int_0^2 4x^4 - 32x^2 + 64\,dx$$

$$2\left(\tfrac{4}{5}x^5 - \tfrac{32}{3}x^3 + 64x \Big|_0^2\right)$$

$$2\left(\frac{128}{5} - \frac{256}{3} + 128\right) = 2\left(\frac{384}{15} - \frac{1280}{15} + \frac{1920}{15}\right) = \frac{2048}{15}$$

b. Find the volume of all of the cross sections perpendicular to the x-axis are semicircles.

$$r = \tfrac{1}{2}(-2x^2+8)$$

$$r = -x^2 + 4$$

$$A(x) = \tfrac{1}{2}\pi r^2$$

$$= \tfrac{1}{2}\pi(-x^2+4)^2$$

$$= \tfrac{1}{2}\pi(x^4 - 8x^2 + 16)$$

$$2\int_0^2 \tfrac{1}{2}\pi(x^4 - 8x^2 + 16)\,dx$$

$$\frac{2\pi}{2}\left[\tfrac{1}{5}x^5 - \tfrac{8}{3}x^3 + 16x \Big|_0^2\right]$$

$$\frac{2\pi}{2}\left(\frac{32}{5} - \frac{64}{3} + 32\right)$$

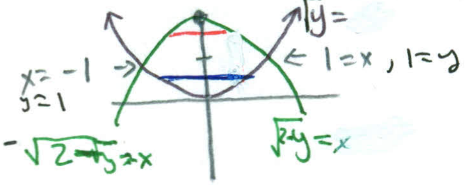
$$= \frac{2\pi}{2}\left(\frac{96}{15} - \frac{320}{15} + \frac{480}{15}\right)$$

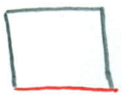
$$\frac{\pi \cdot 256}{15} = \boxed{\frac{256\pi}{15}}$$

$\sqrt{y} = x$

$-\sqrt{y} = x$

5. Find the volume of a solid which lies between the functions $y = x^2$ and $y = 2 - x^2$ whose cross sections are squares perpendicular to the y-axis





$s = \sqrt{2-y} + \sqrt{2-y}$

$= 2\sqrt{2-y}$

$A(x) = 4(2-y) = s^2$

$\int_1^2 8 - 4y\, dy = 8y - 2y^2 \Big|_1^2$

$16 - 8 - 8 + 2 = 2$

$s = 2\sqrt{y}$

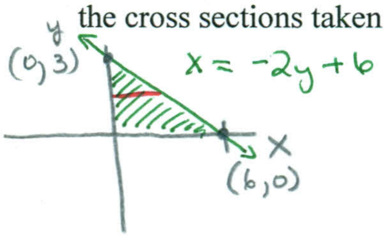
$A(x) = 4y$

$\int_0^1 4y\, dy = 2y^2 \Big|_0^1 = 2$

$V + V$

$2 + 2 = \boxed{4}$

6. A solid has its base is the region bounded by the lines x + 2y = 6, x = 0 and y = 0 and the cross sections taken perpendicular to y-axis are circles. Find the volume the solid.



$d = -2y + 6$

$r = -y + 3$

$A(y) = \pi r^2$

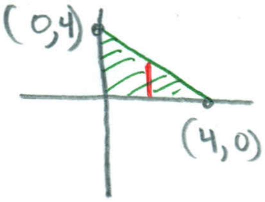
$= \pi(-y+3)^2 = \pi(y^2 - 6y + 9)$

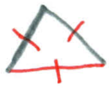
$\int_0^3 \pi(y^2 - 6y + 9)\, dy$

$\pi\left(\frac{1}{3}y^3 - \frac{6}{2}y^2 + 9x \Big|_0^3\right)$

$\pi(9 - 27 + 27) = \boxed{9\pi}$

7. A solid has its base is the region bounded by the lines x + y = 4, x = 0 and y = 0 and the cross section is perpendicular to the x-axis are equilateral triangles. Find its volume.





$s = -x + 4$

$A(x) = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(-x+4)^2$

$= \frac{\sqrt{3}}{4}(x^2 - 8x + 16)$

$\frac{\sqrt{3}}{4}\int_0^4 (x^2 - 8x + 16)\, dx$

$\frac{\sqrt{3}}{4}\left(\frac{1}{3}x^3 - 4x^2 + 16x \Big|_0^4\right)$

$\frac{\sqrt{3}}{4}\left(\frac{64}{3} - 64 + 64\right) = \boxed{\frac{16\sqrt{3}}{3}}$

8. Let the first quadrant region enclosed by the graph of $y = \frac{1}{x}$ and the lines x=1 and x=4 be the base of the solid. If cross sections perpendicular to the x-axis are semicircles, the volume of the solid is

$r = \frac{1}{2x}$

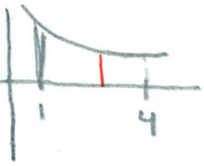
(A) $\frac{3\pi}{64}\,units^3$

(B) $\frac{3\pi}{32}\,units^3$

(C) $\frac{3\pi}{16}\,units^3$

(D) $\frac{3\pi}{8}\,units^3$

(E) $\frac{3\pi}{4}\,units^3$



$A(x) = \frac{1}{2}\pi r^2$

$\int_1^4 \frac{\pi}{8x^2}\,dx$

$\frac{\pi}{8}\int_1^4 x^{-2}\,dx$

$\frac{\pi}{8}\left(-x^{-1}\Big|_1^4\right)$

$\frac{\pi}{8}\left(-\frac{1}{4}-(-1)\right) = \frac{\pi}{8}\left(\frac{3}{4}\right) = \boxed{\frac{3\pi}{32}}$ unit^3

9. Let the base of a solid be the first quadrant region enclosed by the x-axis, the ~~axis~~, the y-axis and the graph of $y = 1 - \frac{x^2}{4}$. If all the cross sections perpendicular to the y-axis are squares, the volume of the solid is

(A) 3 units^3

(B) 2 units^3

(C) 1 units^3

(D) ½ units^3

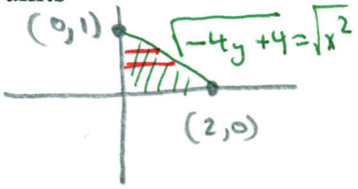
(E) $^1/_3$ units^3

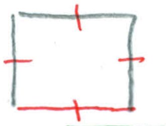
$y = 1 - \frac{x^2}{4}$

$y - 1 = -\frac{x^2}{4}$

$4y - 4 = x^2$

$\sqrt{-4y+4} = \sqrt{x^2}$





$s = \sqrt{-4y+4}$

$A(y) = -4y + 4 = s^2$

$\int_0^1 -4y + 4\,dy$

$-2y^2 + 4y\,\Big|_0^1 = -2 + 4 = \boxed{2}$

10. A solid as its base the region enclosed by the graph of $y = \cos x$ and the x-axis between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. If every cross section perpendicular to the x-axis is a square, the volume of the solid is

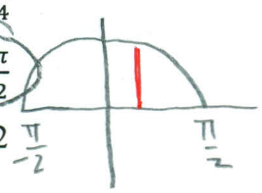
(A) $\frac{\pi}{4}$

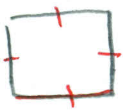
(B) $\frac{\pi^2}{4}$

(C) $\frac{\pi}{2}$

(D) $\frac{\pi^2}{2}$

(E) 2





$s = \cos x$

$A(x) = \cos^2(x) = s^2$

symmetry

$2\int_0^{\frac{\pi}{2}} [\cos(x)]^2\,dx$

s
c
-s
-c

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$= \cos^2(x) - (1 - \cos^2(x))$

$\cos(2x) = 2\cos^2(x) - 1$

$\frac{\cos 2x + 1}{2} = \cos^2(x)$

HOMEWORK: pg 390 2, 4, 8

$\cancel{2}\int_0^{\pi/2} \cancel{\frac{1}{2}}(\cos 2x + 1)\,dx$

$\frac{1}{2}\sin(2x) + x\,\Big|_0^{\pi/2} = \frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}x\right) + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$