Section 3.3 Differentiation Formulas

This section is fundamental to all of calculus. It shows you how to calculate the derivative for many types of functions using formulas rather than using the limit definition.

Concepts to Master

A. The derivatives of $f(x) = c$ and $f(x) = x^n$

B. The derivatives of sums and differences

C. The derivativies of products and quotients

Summary and Focus Questions

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A. The derivative of a constant function $f(x) = c$ is zero.

(**General Power Rule**) For any real number n, the derivative of the function $f(x) = x^n$ is $f(x) = nx^{n-1}$.

In symbols these rules are

$$\frac{d}{dx}(c) = 0 \qquad \text{and} \qquad \frac{d}{dx}(x^n) = nx^{n-1}.$$

1) True, False:

a) For $f(x) = -5, f'(x) = 0$.

True.

b) For $f(x) = x^6, f'(x) = 6x$.

False, $f'(x) = 6x^5$.

c) For $f(x) = 2^x, f'(x) = x2^{x-1}$.

False. The variable x must be in the base of the expression and the exponent must be a constant; we will have to wait a while before we find out that the derivative of 2^x is $(\ln 2)2^x$.

2) Find $f'(x)$ for each:

a) $f(x) = 10$

$f'(x) = 0$.

b) $f(x) = x^{-3}$

$f'(x) = -3x^{-4} = -\frac{3}{x^4}$.

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B. The derivative of a sum or difference of functions, or the multiple of a function is found by putting together the derivatives of the components using these rules:

Name	Rule	Example $H(x)$	$H'(x)$
Constant multiple	$[cf(x)]' = cf'(x)$	$10x^3$	$30x^2\ (= 10(3x^2))$
Sum Rule	$[f(x) + g(x)]' = f'(x) + g'(x)$	$x^3 + x^4$	$3x^2 + 4x^3$
Difference Rule	$[f(x) - g(x)]' = f'(x) - g'(x)$	$x^5 - x^2$	$5x^4 - 2x$

The derivative of a polynomial function is found by applying these rules, perhaps several times.

Example: To find the derivative of $f(x) = 9x^2 - 11x^3 + 7$ we first note that f is the sum or difference of three terms. We find the derivative of each:

$$\frac{d}{dx}(9x^2) = 9\frac{d}{dx}x^2 = 9(2x) = 18x.$$

$$\frac{d}{dx}(11x^3) = 11\frac{d}{dx}x^3 = 11(3x^2) = 33x^2.$$

$$\frac{d}{dx}(7) = 0.$$

Therefore, $f'(x) = 18x - 33x^2 + 0 = 18x - 33x^2$.

You should become very proficient at using these and other differentiation rules as they will be used throughout the text.

3) Find $f'(x)$ for each:

a) $f(x) = 5x^4$

$f'(x) = 20x^3$.

b) $f(x) = x^4 + 4x^3$

$f'(x) = 4x^3 + 12x^2$.

c) $f(x) = 7x^3 + 6x^2 + 10x + 12$

$f'(x) = 21x^2 + 12x + 10$.

d) $f(x) = x^{-7} - x^{-6}$

$f'(x) = -7x^{-8} + 6x^{-7}$.

e) $f(x) = x^{\pi}$

$f'(x) = \pi x^{\pi - 1}$.

f) $f(x) = 3x^3 - 6x^{-2}$

$f'(x) = 9x^2 + 12x^{-3}$

g) $f(x) = (x + 1)^2$

For now we have no way to find the derivative except to first multiply out the expression:

$$f(x) = (x + 1)^2 = x^2 + 2x + 1.$$

Thus, $f'(x) = 2x + 2$.

(We will see another way to deal with these types of function later in this chapter.)

4) Suppose the position of a particle along an axis at time t is $f(t) = t^2 + \frac{1}{t}$ ft. Find the velocity of the particle at time $t = 2$ seconds.

We can now use differentiation formulas to find velocities.

$f(t) = t^2 + t^{-1}$

$f'(t) = 2t + (-1)t^{-2} = 2t - \frac{1}{t^2}$

$f'(2) = 2(2) - \frac{1}{2^2} = 3.75$ ft/s.

5) For $f(x) = mx + b$, find $f'(x)$. What does this say about the tangent line to $f(x)$ at any point?

$f'(x) = m$, a constant. Therefore, the tangent line always has slope m. Since $f(x)$ has slope m, the tangent line at any point is the line itself.

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C. The derivative of a product or quotient of functions is found by putting together the derivatives of the components using these rules. They are not as simple as the sum and difference rules.

Name	Rule	Example $H(x)$	$H'(x)$
Product Rule	$[f(x) \cdot g(x)]' = f(x)g'(x) + f'(x)g(x)$	$x^3 \sin x$	$x^3 \cos x + (\sin x)3x^2$
Quotient Rule	$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$	$\frac{x^4}{x^2+1}$	$\frac{(x^2+1)(4x^3) - x^4(2x)}{(x^2+1)^2}$

6) Find $f'(x)$ for each:

a) $f(x) = 5x^4(x + 1)$

$$f'(x) = 20x^3(x + 1) + 5x^4(1) = 25x^4 + 20x^3.$$

b) $f(x) = (x^2 + x)(3x + 1)$

$$f'(x) = (x^2 + x)(3) + (3x + 1)(2x + 1) = 9x^2 + 8x + 1.$$

c) $f(x) = \dfrac{x^3}{x^2 + 10}$

$f'(x) = \dfrac{(x^2 + 10)(3x^2) - (x^3)(2x)}{(x^2 + 10)^2} = \dfrac{x^4 + 30x^2}{(x^2 + 10)^2}.$

d) $f(x) = \dfrac{1 + x^{-1}}{2 - x^{-2}}$

$f'(x) = \dfrac{(2 - x^{-2})(-x^{-2}) - (1 + x^{-1})(2x^{-3})}{(2 - x^{-2})^2}.$

e) $f(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$

$f(x) = \dfrac{x^{1/2} - 1}{x^{1/2} + 1}$, so $f'(x) =$

$$\frac{(x^{1/2} + 1)\left(\frac{1}{2}x^{-1/2}\right) - (x^{1/2} - 1)\left(\frac{1}{2}x^{-1/2}\right)}{(x^{1/2} + 1)^2}$$

$$= \frac{2\left(\frac{1}{2}x^{-1/2}\right)}{(x^{1/2} + 1)^2} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}.$$

f) $f(x) = x^2e^x$

$f'(x) = x^2(e^x) + e^x(2x) = (x^2 + 2x)e^x.$

7) Find $f'(x)$ for $f(x) = \dfrac{1}{x^2}$ two ways:

a) by the Power Rule

$f(x) = \dfrac{1}{x^2} = x^{-2}.$

Thus, $f'(x) = -2x^{-3} = \dfrac{-2}{x^3}.$

b) by considering $\dfrac{1}{x^2}$ as a quotient.

$f'(x) = \dfrac{x^2(0) - 1(2x)}{(x^2)^2} = \dfrac{-2x}{x^4} = \dfrac{-2}{x^3}.$

8) Find $f'(x)$ for $f(x) = 4x^3$ using the product rule.

$f'(x) = 0 \cdot x^3 + 4 \cdot (3x^2) = 12x^2.$

Note: This problem demonstrates that the constant multiple rule is a special case of the product rule. For problems like this one it is easier to use the constant multiple rule.

Section 3.4 Derivatives of Trigonometric Functions

This section contains formulas for derivatives of another type of function—trigonometric functions. We are continuing to build up a base of basic derivative rules that is useful throughout calculus. Here again, after the basic rules are mastered, you need to use them to find derivatives of more complex functions.

Concepts to Master

A. Limits involving trigonometric functions

B. Derivatives of the six trigonometric functions

Summary and Focus Questions

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A. Two very important limits involving sine and cosine are:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad\text{and}\quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

where x is the radian measure of an angle.

You may use these two basic limits and the limit laws to find limits of more complex functions.

Example: Evaluate $\lim\limits_{t\to 0}\frac{\sin 3t}{4t}$.

Since this resembles $\frac{\sin x}{x}$, we choose $x = 3t$. Then as $t \to 0$, $x \to 0$ and

$$\lim_{t\to 0}\frac{\sin 3t}{4t} = \lim_{t\to 0}\left(\frac{3}{4}\right)\frac{\sin 3t}{3t} = \frac{3}{4}\lim_{x\to 0}\frac{\sin x}{x} = \frac{3}{4}(1) = \frac{3}{4}.$$

1) $\lim\limits_{x\to 0}\frac{1-\cos x}{x} =$ ______.

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0} -\frac{\cos x - 1}{x} = -0 = 0.$$

2) Find $\lim\limits_{x\to 0}\frac{1}{x\cot x} =$ ______.

$$\frac{1}{x\cot x} = \frac{1}{x\frac{\cos x}{\sin x}} = \frac{\sin x}{x\cos x} = \frac{\sin x}{x}\cdot\frac{1}{\cos x}.$$

Then $\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = 1\cdot\frac{1}{1} = 1.$

3) Find $\lim_{x \to 0} \frac{x^2}{2 \sin x}$.

$$\lim_{x \to 0} \frac{x^2}{2 \sin x} = \lim_{x \to 0} \frac{x}{2} \cdot \frac{x}{\sin x}$$

$$= \lim_{x \to 0} \frac{x}{2} \cdot \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \lim_{x \to 0} \frac{x}{2} \cdot \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}}$$

$$= (0)\left(\frac{1}{1}\right) = 0.$$

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B. These six differentiation formulas *must* be memorized.

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x \qquad \frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = (\sec x)(\tan x) \qquad \frac{d}{dx} \csc x = -(\csc x)(\cot x)$$

4) Find y' for each:

a) $y = \sin x - \cos x$

$$y' = \cos x - (-\sin x)$$
$$y' = \cos x + \sin x.$$

b) $y = \frac{\tan x}{x + 1}$

$$y' = \frac{(x + 1)(\sec^2 x) - (\tan x)(1)}{(x + 1)^2}$$
$$= \frac{x \sec^2 x + \sec^2 x - \tan x}{(x + 1)^2}.$$

c) $y = \sin \frac{\pi}{4}$

$y' = 0$, since y is a constant.

d) $y = x^3 \sin x$

$$y' = x^3(\cos x) + (\sin x)3x^2$$
$$= x^3 \cos x + 3x^2 \sin x.$$

e) $y = x^2 + 2x \cos x$

$$y' = 2x + 2x(-\sin x) + 2 \cos x$$
$$= 2(x - x \sin x + \cos x).$$

f) $y = \frac{x}{\sec x + 1}$

$$y' = \frac{(\sec x + 1)1 - x(\sec x \tan x)}{(\sec x + 1)^2}$$
$$= \frac{\sec x + 1 - x \sec x \tan x}{(\sec x + 1)^2}.$$

g) $y = \frac{x}{\cot x}$

<You could evaluate this using the quotient rule but the problem is probably easier if you first rewrite y with a trigonometric identity and then use the product rule.>

First rewrite as $y = x \tan x$.
$y' = x \sec^2 x + \tan x$.

5) Find where the graph of $f(x) = \sqrt{3} \sin x + 3 \cos x$ has a horizontal tangent.

<If the tangent line is horizontal, then its slope ($f'(x)$) is zero. Set $f'(x) = 0$ and solve for x.>

Since $f'(x) = \sqrt{3} \cos x - 3\sin x$, set
$f'(x) = \sqrt{3} \cos x - 3 \sin x = 0$.
$\sqrt{3} \cos x = 3 \sin x$
$\frac{\sin x}{\cos x} = \frac{\sqrt{3}}{3}$; $\tan x = \frac{\sqrt{3}}{3}$.
There are horizontal tangents at $x = \frac{\pi}{6} + k\pi$, where k is an integer.