

x	0	1	2
$f(x)$	1	k	2

26. The function f is continuous on the closed interval $[0, 2]$ and has values that are given in the table above. The equation $f(x) = \frac{1}{2}$ must have at least two solutions in the interval $[0, 2]$ if $k =$
- (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 3
91. Let f be a function that is differentiable on the open interval $(1, 10)$. If $f(2) = -5$, $f(5) = 5$, and $f(9) = -5$, which of the following must be true?
- I. f has at least 2 zeros.
II. The graph of f has at least one horizontal tangent.
III. For some c , $2 < c < 5$, $f(c) = 3$.
- (A) None
(B) I only
(C) I and II only
(D) I and III only
(E) I, II, and III

26. A Any value of k less than $1/2$ will require the function to assume the value of $1/2$ at least twice because of the Intermediate Value Theorem on the intervals $[0,1]$ and $[1,2]$. Hence $k = 0$ is the only option.

91. E I. True. Apply the Intermediate Value Theorem to each of the intervals $[2,5]$ and $[5,9]$.
II. True. Apply the Mean Value Theorem to the interval $[2,9]$.
III. True. Apply the Intermediate Value Theorem to the interval $[2,5]$.

35. If F and f are differentiable functions such that $F(x) = \int_0^x f(t)dt$, and if $F(a) = -2$ and $F(b) = -2$ where $a < b$, which of the following must be true?

(A) $f(x) = 0$ for some x such that $a < x < b$.

(B) $f(x) > 0$ for all x such that $a < x < b$.

(C) $f(x) < 0$ for all x such that $a < x < b$.

(D) $F(x) \leq 0$ for all x such that $a < x < b$.

(E) $F(x) = 0$ for some x such that $a < x < b$.

44. If f is continuous on the interval $[a, b]$, then there exists c such that $a < c < b$ and $\int_a^b f(x)dx =$

(A) $\frac{f(c)}{b-a}$ (B) $\frac{f(b)-f(a)}{b-a}$ (C) $f(b)-f(a)$ (D) $f'(c)(b-a)$ (E) $f(c)(b-a)$

35. A Apply the Mean Value Theorem to F . $F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{0}{a} = 0$. This means that there is number in the interval (a, b) for which F' is zero. However, $F'(x) = f(x)$. So, $f(x) = 0$ for some number in the interval (a, b) .

44. E By the Intermediate Value Theorem there is a c satisfying $a < c < b$ such that $f(c)$ is equal to the average value of f on the interval $[a, b]$. But the average value is also given by $\frac{1}{b - a} \int_a^b f(x) dx$. Equating the two gives option E.

Alternatively, let $F(t) = \int_a^t f(x) dx$. By the Mean Value Theorem, there is a c satisfying

$a < c < b$ such that $\frac{F(b) - F(a)}{b - a} = F'(c)$. But $F(b) - F(a) = \int_a^b f(x) dx$, and $F'(c) = f(c)$ by the Fundamental Theorem of Calculus. This again gives option E as the answer. This result is called the Mean Value Theorem for Integrals.

13. Let f be the function given by $f(x) = x^3 - 3x^2$. What are all values of c that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval $[0, 3]$?

- (A) 0 only (B) 2 only (C) 3 only (D) 0 and 3 (E) 2 and 3

20. Let f be a polynomial function with degree greater than 2. If $a \neq b$ and $f(a) = f(b) = 1$, which of the following must be true for at least one value of x between a and b ?

- I. $f(x) = 0$
II. $f'(x) = 0$
III. $f''(x) = 0$

- (A) None (B) I only (C) II only (D) I and II only (E) I, II, and III

24. If c is the number that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 - 2x^2$ on the interval $0 \leq x \leq 2$, then $c =$

- (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{4}{3}$ (E) 2

13. B $f(0) = 0, f(3) = 0, f'(x) = 3x^2 - 6x$; by the Mean Value Theorem,
 $f'(c) = \frac{f(3) - f(0)}{3} = 0$ for $c \in (0, 3)$.
So, $0 = 3c^2 - 6c = 3c(c - 2)$. The only value in the open interval is 2.

20. C Consider the cases:
I. false if $f(x) = 1$
II. This is true by the Mean Value Theorem
III. false if the graph of f is a parabola with vertex at $x = \frac{a+b}{2}$.
Only II must be true.

24. D $f'(x) = 3x^2 - 4x, f(0) = 0$ and $f(2) = 0$. By the Mean Value Theorem,
 $0 = \frac{f(2) - f(0)}{2 - 0} = f'(c) = 3c^2 - 4c$ for $c \in (0, 2)$. So, $c = \frac{4}{3}$.

30. If a function f is continuous for all x and if f has a relative maximum at $(-1, 4)$ and a relative minimum at $(3, -2)$, which of the following statements must be true?
- (A) The graph of f has a point of inflection somewhere between $x = -1$ and $x = 3$.
 - (B) $f'(-1) = 0$
 - (C) The graph of f has a horizontal asymptote.
 - (D) The graph of f has a horizontal tangent line at $x = 3$.
 - (E) The graph of f intersects both axes.
3. The Mean Value Theorem guarantees the existence of a special point on the graph of $y = \sqrt{x}$ between $(0, 0)$ and $(4, 2)$. What are the coordinates of this point?
- (A) $(2, 1)$
 - (B) $(1, 1)$
 - (C) $(2, \sqrt{2})$
 - (D) $(\frac{1}{2}, \frac{1}{\sqrt{2}})$
 - (E) None of the above
18. Let g be a continuous function on the closed interval $[0, 1]$. Let $g(0) = 1$ and $g(1) = 0$. Which of the following is NOT necessarily true?
- (A) There exists a number h in $[0, 1]$ such that $g(h) \geq g(x)$ for all x in $[0, 1]$.
 - (B) For all a and b in $[0, 1]$, if $a = b$, then $g(a) = g(b)$.
 - (C) There exists a number h in $[0, 1]$ such that $g(h) = \frac{1}{2}$.
 - (D) There exists a number h in $[0, 1]$ such that $g(h) = \frac{3}{2}$.
 - (E) For all h in the open interval $(0, 1)$, $\lim_{x \rightarrow h} g(x) = g(h)$.

30. E Because f is continuous for all x , the Intermediate Value Theorem implies that the graph of f must intersect the x -axis. The graph must also intersect the y -axis since f is defined for all x , in particular, at $x = 0$.

3. B $y = \sqrt{x}$, $y' = \frac{1}{2\sqrt{x}}$. By the Mean Value Theorem we have $\frac{1}{2\sqrt{c}} = \frac{2}{4} \Rightarrow c = 1$.

The point is (1,1).

18. D D could be false, consider $g(x) = 1 - x$ on $[0,1]$. A is true by the Extreme Value Theorem, B is true because g is a function, C is true by the Intermediate Value Theorem, and E is true because g is continuous.

No Calculator

19. Let $f(x)$ be a differentiable function defined only on the interval $-2 \leq x \leq 10$. The table below gives the value of $f(x)$ and its derivative $f'(x)$ at several points of the domain.

x	-2	0	2	4	6	8	10
$f(x)$	26	27	26	23	18	11	2
$f'(x)$	1	0	-1	-2	-3	-4	-5

The line tangent to the graph of $f(x)$ and parallel to the segment between the endpoints intersects the y -axis at the point

- (A) (0, 27)
- (B) (0, 28)
- (C) (0, 31)
- (D) (0, 36)
- (E) (0, 43)

19. The segment with endpoints $(-2, 26)$ and $(10, 2)$ has a slope of -2 . Therefore, by the Mean Value Theorem, the line through $(4, 23)$ will be parallel to the segment and tangent to the graph; its equation is $y - 23 = -2(x - 4)$. Its y -intercept is 31.

The correct choice is (C).

Calculator Active

5. Given: $f(t) = 2\pi t + \sin(2\pi t)$

- (a) Find a number t in the open interval $(0, 2)$ that satisfies the conclusion of the Mean Value Theorem on the interval $[0, 2]$.
- (b) Suppose the given function describes the position of a particle on the x -axis for time $0 \leq t \leq 2$. What is the average velocity of the particle over that interval?
- (c) Determine the velocity and the acceleration of the particle at $t = 1$.

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(a) $f(t) = 2\pi t + \sin(2\pi t)$

The Mean Value Theorem concludes that (under suitable conditions satisfied by the function $f(t)$ on the interval $[a, b]$), there will be a t -coordinate at which

$$f'(t) = \frac{f(b) - f(a)}{b - a}.$$

For this function f on $[0, 2]$,

$$\frac{f(2) - f(0)}{2 - 0} = \frac{(4\pi + \sin 4\pi) - (0 + \sin 0)}{2} = 2\pi.$$

In addition, $f'(t) = 2\pi + 2\pi \cos(2\pi t)$.

We set these last two results equal to each other.

$$\begin{aligned} 2\pi + 2\pi \cos(2\pi t) &= 2\pi &\Rightarrow &\cos(2\pi t) = 0 \\ &&\Rightarrow &2\pi t = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}. \\ &&\Rightarrow &t = \frac{1}{4} \text{ or } \frac{3}{4} \text{ or } \frac{5}{4} \text{ or } \frac{7}{4}. \end{aligned}$$

Any one of these is a suitable t .

(b) Average velocity = $\frac{\text{change in position}}{\text{change in time}} = \frac{f(2) - f(0)}{2 - 0} = 2\pi.$

This was calculated above.

$$\begin{aligned} \text{(c) } v(t) = f'(t) &= 2\pi + 2\pi \cos(2\pi t) &\Rightarrow &v(1) = 2\pi + 2\pi = 4\pi \\ a(t) = f''(x) &= -4\pi^2 \sin(2\pi t) &\Rightarrow &a(1) = -4\pi^2 \sin(2\pi) = 0 \end{aligned}$$

44. If f is continuous on the interval $[a, b]$, then there exists c such that $a < c < b$ and $\int_a^b f(x)dx =$

- (A) $\frac{f(c)}{b-a}$ (B) $\frac{f(b)-f(a)}{b-a}$ (C) $f(b)-f(a)$ (D) $f'(c)(b-a)$ (E) $f(c)(b-a)$

20. The average value of $\cos x$ on the interval $[-3, 5]$ is

- (A) $\frac{\sin 5 - \sin 3}{8}$
(B) $\frac{\sin 5 - \sin 3}{2}$
(C) $\frac{\sin 3 - \sin 5}{2}$
(D) $\frac{\sin 3 + \sin 5}{2}$
(E) $\frac{\sin 3 + \sin 5}{8}$

34. The average value of \sqrt{x} over the interval $0 \leq x \leq 2$ is

- (A) $\frac{1}{3}\sqrt{2}$ (B) $\frac{1}{2}\sqrt{2}$ (C) $\frac{2}{3}\sqrt{2}$ (D) 1 (E) $\frac{4}{3}\sqrt{2}$

44. E By the Intermediate Value Theorem there is a c satisfying $a < c < b$ such that $f(c)$ is equal to the average value of f on the interval $[a, b]$. But the average value is also given by $\frac{1}{b-a} \int_a^b f(x) dx$. Equating the two gives option E.

Alternatively, let $F(t) = \int_a^t f(x) dx$. By the Mean Value Theorem, there is a c satisfying $a < c < b$ such that $\frac{F(b) - F(a)}{b - a} = F'(c)$. But $F(b) - F(a) = \int_a^b f(x) dx$, and $F'(c) = f(c)$ by the Fundamental Theorem of Calculus. This again gives option E as the answer. This result is called the Mean Value Theorem for Integrals.

20. E $\frac{1}{8} \int_{-3}^5 \cos x dx = \frac{1}{8} (\sin 5 - \sin(-3)) = \frac{1}{8} (\sin 5 + \sin 3)$; Note: Since the sine is an odd function, $\sin(-3) = -\sin(3)$.

$$34. C \quad \frac{1}{2} \int_0^2 \sqrt{x} dx = \frac{1}{2} \cdot \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 = \frac{1}{3} \cdot 2^{\frac{3}{2}} = \frac{2}{3} \sqrt{2}$$

36. What is the average value of y for the part of the curve $y = 3x - x^2$ which is in the first quadrant?

- (A) -6 (B) -2 (C) $\frac{3}{2}$ (D) $\frac{9}{4}$ (E) $\frac{9}{2}$

44. The average value of $f(x) = x^2\sqrt{x^3+1}$ on the closed interval $[0,2]$ is

- (A) $\frac{26}{9}$ (B) $\frac{13}{3}$ (C) $\frac{26}{3}$ (D) 13 (E) 26

21. The average value of $\frac{1}{x}$ on the closed interval $[1,3]$ is

- (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\ln 2}{2}$ (D) $\frac{\ln 3}{2}$ (E) $\ln 3$

33. What is the average (mean) value of $3t^3 - t^2$ over the interval $-1 \leq t \leq 2$?

- (A) $\frac{11}{4}$ (B) $\frac{7}{2}$ (C) 8 (D) $\frac{33}{4}$ (E) 16

36. C $3x - x^2 = x(3 - x) > 0$ for $0 < x < 3$

$$\text{Average value} = \frac{1}{3} \int_0^3 (3x - x^2) dx = \frac{1}{3} \left(\frac{3}{2} x^2 - \frac{1}{3} x^3 \right) \Big|_0^3 = \frac{3}{2}$$

44. A $\frac{1}{2} \int_0^2 x^2 (x^3 + 1)^{1/2} dx = \frac{1}{2} \cdot \frac{1}{3} \int_0^2 (x^3 + 1)^{1/2} (3x^2 dx) = \frac{1}{6} (x^3 + 1)^{3/2} \cdot \frac{2}{3} \Big|_0^2 = \frac{26}{9}$

21. D $\frac{1}{2} \int_1^3 \frac{1}{x} dx = \frac{1}{2} \ln x \Big|_1^3 = \frac{1}{2} (\ln 3 - \ln 1) = \frac{1}{2} \ln 3$

33. A $\frac{1}{3} \int_{-1}^2 3t^3 - t^2 dt = \frac{1}{3} \left(\frac{3}{4} t^4 - \frac{1}{3} t^3 \right) \Big|_{-1}^2 = \frac{1}{3} \left(\left(12 - \frac{8}{3} \right) - \left(\frac{3}{4} + \frac{1}{3} \right) \right) = \frac{11}{4}$