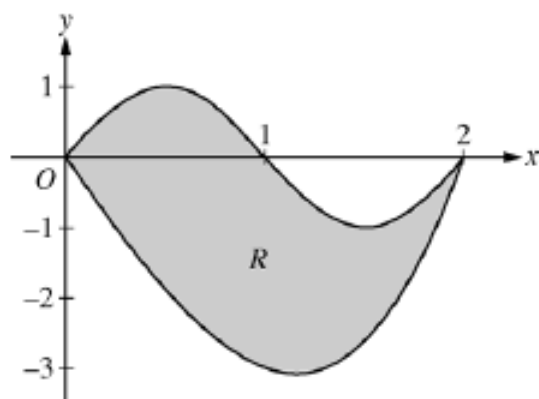


PRACTICE TEST 4 FRQ #1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

- Find the area of R .
- The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.
- The region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is a square. Find the volume of this solid.
- The region R models the surface of a small pond. At all points in R at a distance x from the y -axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$

$$\text{Area} = \int_0^2 (\sin(\pi x) - (x^3 - 4x)) dx = 4$$

$$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$

$$\text{The area of the stated region is } \int_r^s (-2 - (x^3 - 4x)) dx$$

$$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$$

(c) $\text{Volume} = \int_0^2 (\sin(\pi x) - (x^3 - 4x))^2 dx = 9.978$

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(d) $\text{Volume} = \int_0^2 (3 - x)(\sin(\pi x) - (x^3 - 4x)) dx = 8.369$ or 8.370

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

PRACTICE TEST 4 FRQ #2

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin(\sqrt{t+10})$ for $0 \leq t \leq 120$ minutes.

- (a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.
- (b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.
- (c) The scientist proposes the function f , given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.
- (d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \leq t \leq 60$ minutes. Does this value indicate that the water must be diverted?

$$(a) \frac{(0+7)}{2} \cdot 8 + \frac{(7+8)}{2} \cdot 6 + \frac{(8+2)}{2} \cdot 8 + \frac{(2+0)}{2} \cdot 2 = 115 \text{ ft}^2$$

$$(b) \frac{1}{120} \int_0^{120} 115v(t) dt = 1807.169 \text{ or } 1807.170 \text{ ft}^3/\text{min}$$

$$(c) \int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx = 122.230 \text{ or } 122.231 \text{ ft}^2$$

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is

$$\frac{1}{20} \int_{40}^{60} C \cdot v(t) dt = 2181.912 \text{ or } 2181.913 \text{ ft}^3/\text{min}.$$

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds $2100 \text{ ft}^3/\text{min}$.

1 : trapezoidal approximation

3 : $\left\{ \begin{array}{l} 1 : \text{limits and average value} \\ \quad \text{constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \text{integral} \\ 1 : \text{answer} \end{array} \right.$

3 : $\left\{ \begin{array}{l} 1 : \text{volumetric flow integral} \\ 1 : \text{average volumetric flow} \\ 1 : \text{answer with reason} \end{array} \right.$

PRACTICE TEST 4 FRQ #3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

- (a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?
- (b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.
- (c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

- (a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000$ cm³/min
and $\frac{dr}{dt} = 2.5$ cm/min.

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm/min}$$

- (b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$.

This occurs when $t = 25$ minutes.

Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$,

the oil slick reaches its maximum volume 25 minutes after the device begins working.

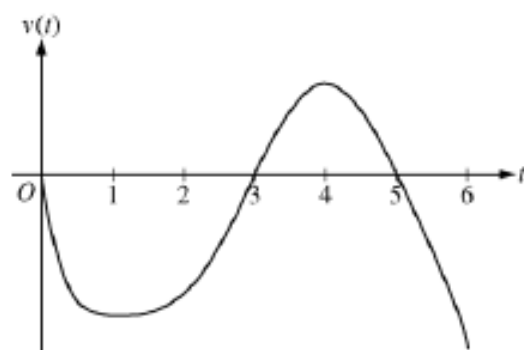
- (c) The volume of oil, in cm³, in the slick at time $t = 25$ minutes is given by $60,000 + \int_0^{25} (2000 - R(t)) dt$.

$$4 : \begin{cases} 1 : \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2 : \text{expression for } \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : R(t) = 2000 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

$$2 : \begin{cases} 1 : \text{limits and initial condition} \\ 1 : \text{integrand} \end{cases}$$

PRACTICE TEST 4 FRQ #4



Graph of v

A particle moves along the x -axis so that its velocity at time t , for $0 \leq t \leq 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t -axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

- (a) For $0 \leq t \leq 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.
- (b) For how many values of t , where $0 \leq t \leq 6$, is the particle at $x = -8$? Explain your reasoning.
- (c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.
- (d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

- (a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$.

$$x(3) = -2 + \int_0^3 v(t) dt = -2 - 8 = -10$$

$$x(6) = -2 + \int_0^6 v(t) dt = -2 - 8 + 3 - 2 = -9$$

Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.

- (b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$.

By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.

- (c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.
- (d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.

$$3 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{considers } \int_0^6 v(t) dt \\ 1 : \text{conclusion} \end{cases}$$

$$3 : \begin{cases} 1 : \text{positions at } t = 3, t = 5, \\ \text{and } t = 6 \\ 1 : \text{description of motion} \\ 1 : \text{conclusion} \end{cases}$$

1 : answer with reason

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

PRACTICE TEST 4 FRQ #5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

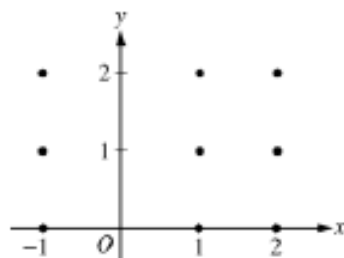
- (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)

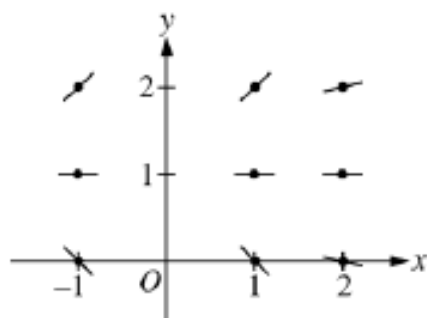
- (b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

- (c) For the particular solution $y = f(x)$ described in part (b), find

$$\lim_{x \rightarrow \infty} f(x).$$



- (a)



$$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$$

(b) $\frac{1}{y-1} dy = \frac{1}{x^2} dx$

$$\ln|y-1| = -\frac{1}{x} + C$$

$$|y-1| = e^{-\frac{1}{x} + C}$$

$$|y-1| = e^C e^{-\frac{1}{x}}$$

$$y-1 = ke^{-\frac{1}{x}}, \text{ where } k = \pm e^C$$

$$-1 = ke^{-\frac{1}{2}}$$

$$k = -e^{\frac{1}{2}}$$

$$f(x) = 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)}, x > 0$$

$$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(c) $\lim_{x \rightarrow \infty} 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

PRACTICE TEST 4 FRQ #6

Let f be the function given by $f(x) = \frac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

- (a) Write an equation for the line tangent to the graph of f at $x = e^2$.
 (b) Find the x -coordinate of the critical point of f . Determine whether this point is a relative minimum, a relative maximum, or neither for the function f . Justify your answer.
 (c) The graph of the function f has exactly one point of inflection. Find the x -coordinate of this point.
 (d) Find $\lim_{x \rightarrow 0^+} f(x)$.

(a) $f(e^2) = \frac{\ln e^2}{e^2} = \frac{2}{e^2}$, $f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = -\frac{1}{e^4}$

An equation for the tangent line is $y = \frac{2}{e^2} - \frac{1}{e^4}(x - e^2)$.

2 : $\begin{cases} 1 : f(e^2) \text{ and } f'(e^2) \\ 1 : \text{answer} \end{cases}$

- (b) $f'(x) = 0$ when $x = e$. The function f has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.

3 : $\begin{cases} 1 : x = e \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$

(c) $f''(x) = \frac{-\frac{1}{x}x^2 - (1 - \ln x)2x}{x^4} = \frac{-3 + 2 \ln x}{x^3}$ for all $x > 0$

$f''(x) = 0$ when $-3 + 2 \ln x = 0$

$x = e^{3/2}$

The graph of f has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.

3 : $\begin{cases} 2 : f''(x) \\ 1 : \text{answer} \end{cases}$

(d) $\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty$ or Does Not Exist

1 : answer