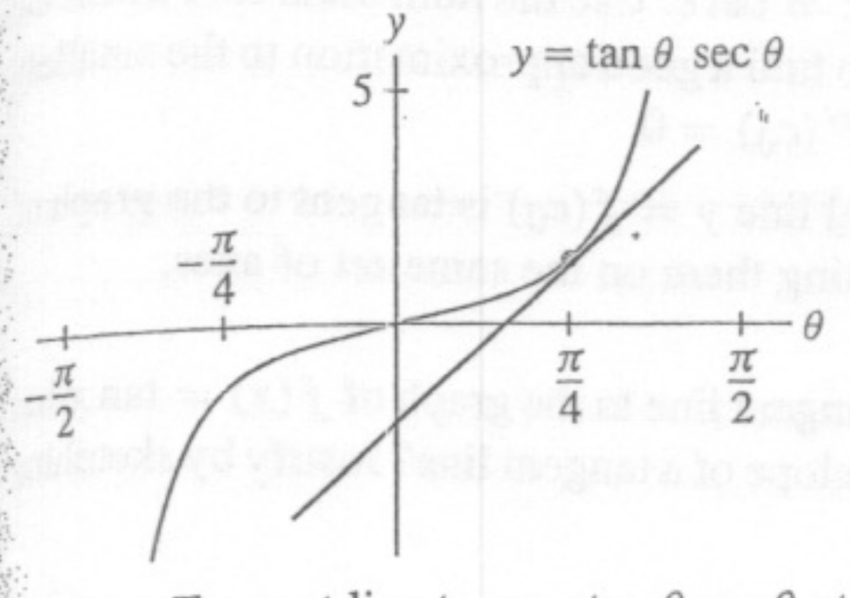


FIGURE 3 Tangent line to $y = \tan\theta\sec\theta$ at $\theta = \frac{\pi}{4}$.

Now use the values $\sec\frac{\pi}{4} = \sqrt{2}$ and $\tan\frac{\pi}{4} = 1$ to compute

$$y\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$y'\left(\frac{\pi}{4}\right) = \tan^2\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) + \sec^3\left(\frac{\pi}{4}\right) = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

An equation of the tangent line (Figure 3) is $y - \sqrt{2} = 3\sqrt{2}\left(\theta - \frac{\pi}{4}\right)$. ■

3.6 SUMMARY

- Basic trigonometric derivatives:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x$$

- Additional formulas:

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

3.6 EXERCISES

Preliminary Questions

1. Determine the sign (+ or −) that yields the correct formula for the following:

(a) $\frac{d}{dx}(\sin x + \cos x) = \pm\sin x \pm \cos x$

(b) $\frac{d}{dx}\sec x = \pm\sec x\tan x$

(c) $\frac{d}{dx}\cot x = \pm\csc^2 x$

2. Which of the following functions can be differentiated using the rules we have covered so far?

(a) $y = 3\cos x\cot x$ (b) $y = \cos(x^2)$ (c) $y = e^x\sin x$

3. Compute $\frac{d}{dx}(\sin^2 x + \cos^2 x)$ without using the derivative formulas for $\sin x$ and $\cos x$.

4. How is the addition formula used in deriving the formula $(\sin x)' = \cos x$?

Exercises

In Exercises 1–4, find an equation of the tangent line at the point indicated.

1. $y = \sin x, \quad x = \frac{\pi}{4}$
2. $y = \cos x, \quad x = \frac{\pi}{3}$
3. $y = \tan x, \quad x = \frac{\pi}{4}$
4. $y = \sec x, \quad x = \frac{\pi}{6}$

In Exercises 5–24, compute the derivative.

5. $f(x) = \sin x\cos x$
6. $f(x) = x^2\cos x$
7. $f(x) = \sin^2 x$
8. $f(x) = 9\sec x + 12\cot x$
9. $H(t) = \sin t\sec^2 t$
10. $h(t) = 9\csc t + t\cot t$
11. $f(\theta) = \tan\theta\sec\theta$
12. $k(\theta) = \theta^2\sin^2\theta$
13. $f(x) = (2x^4 - 4x^{-1})\sec x$
14. $f(z) = z\tan z$
15. $y = \dfrac{\sec\theta}{\theta}$
16. $G(z) = \dfrac{1}{\tan z - \cot z}$
17. $R(y) = \dfrac{3\cos y - 4}{\sin y}$
18. $f(x) = \dfrac{x}{\sin x + 2}$
19. $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$
20. $f(\theta) = \theta\tan\theta\sec\theta$
21. $f(x) = e^x\sin x$
22. $h(t) = e^t\csc t$
23. $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$
24. $f(x) = xe^x\cos x$

In Exercises 25–34, find an equation of the tangent line at the point specified.

25. $y = x^3 + \cos x, \quad x = 0$

26. $y = \tan\theta, \quad \theta = \frac{\pi}{6}$

27. $y = \sin x + 3\cos x, \quad x = 0$

28. $y = \dfrac{\sin t}{1+\cos t}, \quad t = \frac{\pi}{3}$

29. $y = 2(\sin\theta + \cos\theta), \quad \theta = \frac{\pi}{3}$

30. $y = \csc x - \cot x, \quad x = \frac{\pi}{4}$

31. $y = e^x \cos x, \quad x = 0$

32. $y = e^x \cos^2 x, \quad x = \frac{\pi}{4}$

33. $y = e^t(1 - \cos t), \quad t = \frac{\pi}{2}$

34. $y = e^\theta \sec\theta, \quad \theta = \frac{\pi}{4}$

In Exercises 35–37, use Theorem 1 to verify the formula.

35. $\dfrac{d}{dx}\cot x = -\csc^2 x$

36. $\dfrac{d}{dx}\sec x = \sec x \tan x$

37. $\dfrac{d}{dx}\csc x = -\csc x \cot x$

38. Show that both $y = \sin x$ and $y = \cos x$ satisfy $y'' = -y$.

In Exercises 39–42, calculate the higher derivative.

39. $f''(\theta), \quad f(\theta) = \theta\sin\theta$

40. $\dfrac{d^2}{dt^2}\cos^2 t$

41. $y'', \quad y''', \quad y = \tan x$

42. $y'', \quad y''', \quad y = e^t \sin t$

43. Calculate the first five derivatives of $f(x) = \cos x$. Then determine $f^{(8)}$ and $f^{(37)}$.

44. Find $y^{(157)}$, where $y = \sin x$.

45. Find the values of x between 0 and 2π where the tangent line to the graph of $y = \sin x \cos x$ is horizontal.

46. GU Plot the graph $f(\theta) = \sec\theta + \csc\theta$ over $[0, 2\pi]$ and determine the number of solutions to $f'(\theta) = 0$ in this interval graphically. Then compute $f'(\theta)$ and find the solutions.

47. GU Let $g(t) = t - \sin t$.

(a) Plot the graph of g with a graphing utility for $0 \le t \le 4\pi$.

(b) Show that the slope of the tangent line is nonnegative. Verify this on your graph.

(c) For which values of t in the given range is the tangent line horizontal?

48. CAS Let $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$.

(a) Plot $f(x)$ on $[-3\pi, 3\pi]$.

(b) Show that $f'(c) = 0$ if $c = \tan c$. Use the numerical root finder on a computer algebra system to find a good approximation to the smallest *positive* value c_0 such that $f'(c_0) = 0$.

(c) Verify that the horizontal line $y = f(c_0)$ is tangent to the graph of $y = f(x)$ at $x = c_0$ by plotting them on the same set of axes.

49. Show that no tangent line to the graph of $f(x) = \tan x$ has zero slope. What is the least slope of a tangent line? Justify by sketching the graph of $(\tan x)'$.

50. The height at time t (in seconds) of a mass, oscillating at the end of a spring, is $s(t) = 300 + 40\sin t$ cm. Find the velocity and acceleration at $t = \frac{\pi}{3}$ s.

51. The horizontal range R of a projectile launched from ground level at an angle θ and initial velocity v_0 m/s is $R = (v_0^2/9.8)\sin\theta\cos\theta$. Calculate $dR/d\theta$. If $\theta = 7\pi/24$, will the range increase or decrease if the angle is increased slightly? Base your answer on the sign of the derivative.

52. Show that if $\frac{\pi}{2} < \theta < \pi$, then the distance along the x-axis between θ and the point where the tangent line intersects the x-axis is equal to $|\tan\theta|$ (Figure 4).

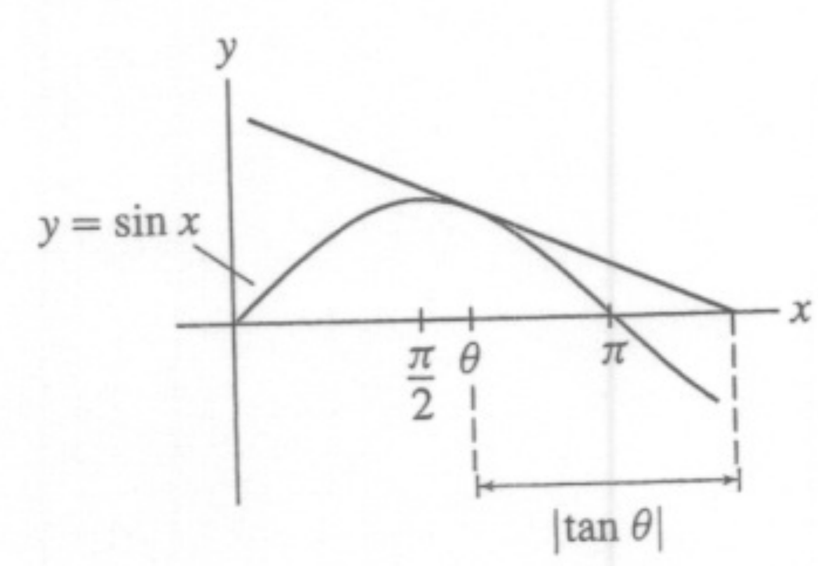


FIGURE 4

Further Insights and Challenges

53. Use the limit definition of the derivative and the addition law for the cosine function to prove that $(\cos x)' = -\sin x$.

54. Use the addition formula for the tangent

$$\tan(x+h) = \frac{\tan x + \tan h}{1 - \tan x \tan h}$$

to compute $(\tan x)'$ directly as a limit of the difference quotients. You will also need to show that $\lim_{h\to 0} \dfrac{\tan h}{h} = 1$.

55. Verify the following identity and use it to give another proof of the formula $(\sin x)' = \cos x$.

$$\sin(x+h) - \sin x = 2\cos\left(x + \tfrac{1}{2}h\right)\sin\left(\tfrac{1}{2}h\right)$$

Hint: Use the addition formula to prove that $\sin(a+b) - \sin(a-b) = 2\cos a \sin b$.

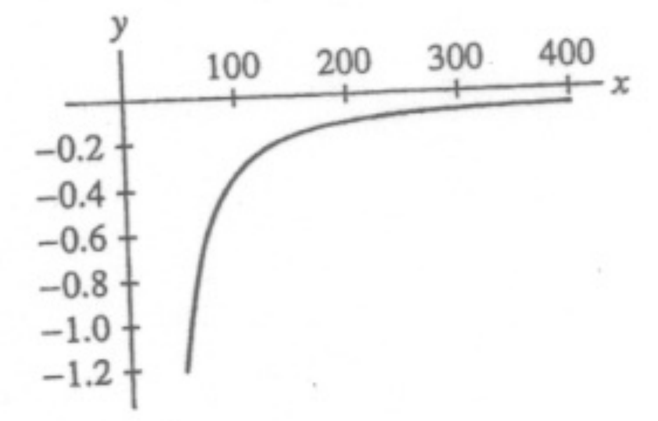
56. Show that a nonzero polynomial function $y = f(x)$ *cannot* satisfy the equation $y'' = -y$. Use this to prove that neither $\sin x$ nor $\cos x$ is a polynomial. Can you think of another way to reach this conclusion by considering limits as $x \to \infty$?

57. Let $f(x) = x\sin x$ and $g(x) = x\cos x$.

(a) Show that $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Verify that $f''(x) = -f(x) + 2\cos x$ and $g''(x) = -g(x) - 2\sin x$.

(c) dS/dQ is plotted below. The fact that this graph is increasing shows that $d^2S/dQ^2 > 0$.



51.
$$f'(x) = -\frac{3}{(x-1)^2} = (-1)^1\frac{3\cdot 1}{(x-1)^{1+1}};$$
$$f''(x) = \frac{6}{(x-1)^3} = (-1)^2\frac{3\cdot 2\cdot 1}{(x-1)^{2+1}};$$
$$f'''(x) = -\frac{18}{(x-1)^4} = (-1)^3\frac{3\cdot 3!}{(x-1)^{3+1}}; \text{ and}$$
$$f^{(4)}(x) = \frac{72}{(x-1)^5} = (-1)^4\frac{3\cdot 4!}{(x-1)^{4+1}}.$$

From the pattern observed above, we conjecture
$$f^{(k)}(x) = (-1)^k\frac{3\cdot k!}{(x-1)^{k+1}}.$$

53. 99!

55. $(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$;
$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}g^{(k)}$$

Section 3.6 Preliminary Questions

1. (a) $\frac{d}{dx}(\sin x + \cos x) = -\sin x + \cos x$

(b) $\frac{d}{dx}\sec x = \sec x\tan x$

(c) $\frac{d}{dx}\cot x = -\csc^2 x$

2. (a) This function can be differentiated using the Product Rule.

(b) We have not yet discussed how to differentiate a function like this.

(c) This function can be differentiated using the Product Rule.

3. 0

4. The difference quotient for the function $\sin x$ involves the expression $\sin(x+h)$. The addition formula for the sine function is used to expand this expression as $\sin(x+h) = \sin x\cos h + \sin h\cos x$.

Section 3.6 Exercises

1. $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)$ **3.** $y = 2x + 1 - \frac{\pi}{2}$

5. $f'(x) = -\sin^2 x + \cos^2 x$ **7.** $f'(x) = 2\sin x\cos x$

9. $H'(t) = 2\sin t\sec^2 t\tan t + \sec t$

11. $f'(\theta) = \left(\tan^2\theta + \sec^2\theta\right)\sec\theta$

13. $f'(x) = (2x^4 - 4x^{-1})\sec x\tan x + \sec x(8x^3 + 4x^{-2})$

15. $y' = \frac{\theta\sec\theta\tan\theta - \sec\theta}{\theta^2}$ **17.** $R'(y) = \frac{4\cos y - 3}{\sin^2 y}$

19. $f'(x) = \frac{2\sec^2 x}{(1-\tan x)^2}$ **21.** $f'(x) = e^x(\cos x + \sin x)$

23. $f'(\theta) = e^\theta(5\sin\theta + 5\cos\theta - 4\tan\theta - 4\sec^2\theta)$

25. $y = 1$ **27.** $y = x + 3$

29. $y = (1-\sqrt{3})\left(x - \frac{\pi}{3}\right) + 1 + \sqrt{3}$

31. $y = x + 1$ **33.** $y = 2e^{\pi/2}\left(t - \frac{\pi}{2}\right) + e^{\pi/2}$

35. $\cot x = \frac{\cos x}{\sin x}$; use the quotient rule

37. $\csc x = \frac{1}{\sin x}$; use the quotient rule

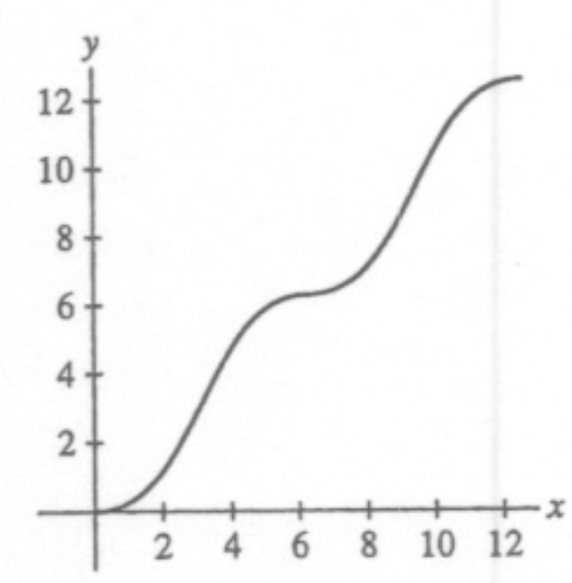
39. $f''(\theta) = -\theta\sin\theta + 2\cos\theta$

41.
$$y'' = 2\sec^2 x\tan x$$
$$y''' = 2\sec^4 x + 4\sec^2 x\tan^2 x.$$

43.
- Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, and $f^{(5)}(x) = -\sin x$.
- Accordingly, the successive derivatives of f cycle among
$$\{-\sin x, -\cos x, \sin x, \cos x\}$$
in that order. Since 8 is a multiple of 4, we have $f^{(8)}(x) = \cos x$.
- Since 36 is a multiple of 4, we have $f^{(36)}(x) = \cos x$. Therefore, $f^{(37)}(x) = -\sin x$.

45. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

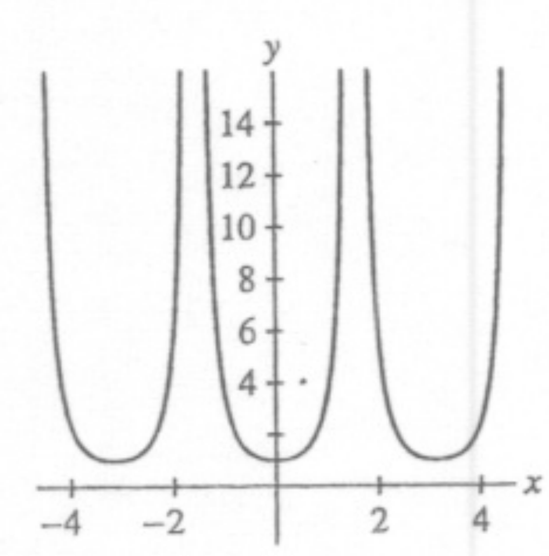
47. (a)



(b) Since $g'(t) = 1 - \cos t \ge 0$ for all t, the slope of the tangent line to g is always nonnegative.

(c) $t = 0, 2\pi, 4\pi$

49. $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$. Note that $f'(x) = \frac{1}{\cos^2 x}$ has numerator 1; the equation $f'(x) = 0$ therefore has no solution. The least slope for a tangent line to $\tan x$ is 1. Here is a graph of f'.



51. $\frac{dR}{d\theta} = (v_0^2/9.8)(-\sin^2\theta + \cos^2\theta)$; if $\theta = 7\pi/24$, increasing the angle will *decrease* the range.