Basic Constructions

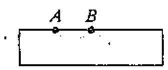
Objectives

1. Perform seven basic constructions.
2. Use these basic constructions in original construction exercises.
3. State and apply theorems involving concurrent lines.

10-1 What Construction Means

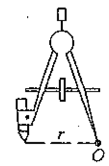
In Chapters 1–9 we have used rulers and protractors to draw segments with certain lengths and angles with certain measures. In this chapter we will *construct* geometric figures using only two instruments, a *straightedge* and a *compass*. (You may use a ruler as a straightedge as long as you do not use the marks on the ruler.)

Using a Straightedge in Constructions



Given two points A and B, we know from Postulate 6 that there is exactly one line through A and B. We agree that we can use a straightedge to draw $\overleftrightarrow{AB}$ or parts of the line, such as $\overline{AB}$ and $\overrightarrow{AB}$.

Using a Compass in Constructions



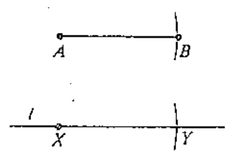
Given a point O and a length r, we know from the definition of a circle that there is exactly one circle with center O and radius r. We agree that we can use a compass to draw this circle or arcs of the circle.

Construction 1

Given a segment, construct a segment congruent to the given segment.

Given: $\overline{AB}$

Construct: A segment congruent to $\overline{AB}$



Procedure:

1. Use a straightedge to draw a line. Call it l.
2. Choose any point on l and label it X.
3. Set your compass for radius AB. Using X as center, draw an arc intersecting line l. Label the point of intersection Y.

$\overline{XY}$ is congruent to $\overline{AB}$.

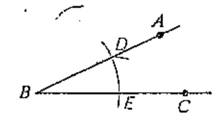
Justification: Since you used AB for the radius of $\odot X$, $\overline{XY} \cong \overline{AB}$.

Construction 2

Given an angle, construct an angle congruent to the given angle.

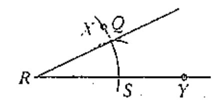
Given: $\angle ABC$

Construct: An angle congruent to $\angle ABC$



Procedure:

1. Draw a ray. Label it $\overrightarrow{RY}$.
2. Using B as center and any radius, draw an arc intersecting $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Label the points of intersection D and E, respectively.
3. Using R as center and the same radius as in Step 2, draw an arc intersecting $\overrightarrow{RY}$. Label the arc $\widehat{XS}$, with S the point where the arc intersects $\overrightarrow{RY}$.
4. Using S as center and a radius equal to DE, draw an arc that intersects $\widehat{XS}$ at a point Q.
5. Draw $\overrightarrow{RQ}$.



$\angle QRS$ is congruent to $\angle ABC$.

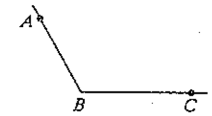
Justification: If you draw $\overline{DE}$ and $\overline{QS}$, $\triangle DBE \cong \triangle QRS$ (SSS Postulate). Then $\angle QRS \cong \angle ABC$.

Construction 3

Given an angle, construct the bisector of the angle.

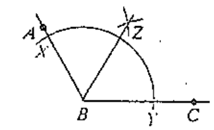
Given: $\angle ABC$

Construct: The bisector of $\angle ABC$



Procedure:

1. Using B as center and any radius, draw an arc that intersects $\overrightarrow{BA}$ at X and $\overrightarrow{BC}$ at Y.
2. Using X as center and a suitable radius, draw an arc. Using Y as center and the same radius, draw an arc that intersects the arc with center X at a point Z.
3. Draw $\overrightarrow{BZ}$.

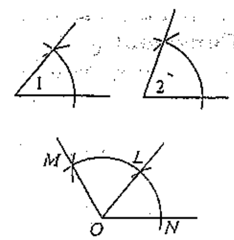


$\overrightarrow{BZ}$ bisects $\angle ABC$.

Justification: If you draw $\overline{XZ}$ and $\overline{YZ}$, $\triangle XBZ \cong \triangle YBZ$ (SSS Postulate). Then $\angle XBZ \cong \angle YBZ$ and $\overrightarrow{BZ}$ bisects $\angle ABC$.

Example Given $\angle 1$ and $\angle 2$, construct an angle whose measure is equal to $m\angle 1 + m\angle 2$.

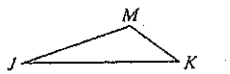
Solution First use Construction 2 to construct $\angle LON$ congruent to $\angle 1$. Then use the same method to construct $\angle MOL$ congruent to $\angle 2$ (as shown) so that $m\angle MON = m\angle 1 + m\angle 2$.



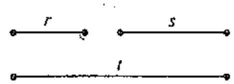
In construction exercises, you won't ordinarily have to write out the procedure and the justification. However, you should be able to supply them when asked to do so.

Classroom Exercises

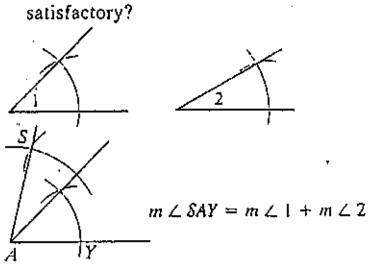
1. Given: $\triangle JKM$
   Explain how to construct a triangle that is congruent to $\triangle JKM$.

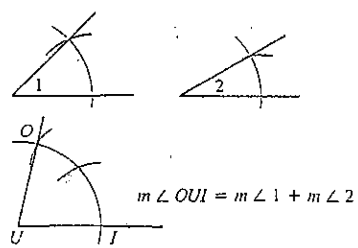


2. Draw any $\overline{AB}$.
   a. Construct $\overline{XY}$ so that $XY = AB$.
   b. Using X and Y as centers, and a radius equal to AB, draw arcs that intersect. Label the point of intersection Z.
   c. Draw $\overline{XZ}$ and $\overline{YZ}$.
   d. What kind of triangle is $\triangle XYZ$?
3. Explain how you could construct a 30° angle.
4. Exercise 3 suggests that you could construct other angles with certain measures. Name some.
5. Suppose you are given the three lengths shown and are asked to construct a triangle whose sides have lengths r, s, and t. Can you do so? State the theorem from Chapter 6 that applies.



6. $\angle 1$ and $\angle 2$ are given. You see two attempts at constructing an angle whose measure is equal to $m\angle 1 + m\angle 2$. Are both constructions satisfactory?





Written Exercises

On your paper, draw two segments roughly like those shown. Use these segments in Exercises 1–4 to construct a segment having the indicated length.

A

1. $a + b$ 2. $b - a$ 3. $3a - b$ 4. $a + 2b$
5. Using any convenient length for a side, construct an equilateral triangle.
6. a. Construct a 30° angle. b. Construct a 15° angle.
7. Draw any acute $\triangle ACU$. Use a method based on the SSS Postulate to construct a triangle congruent to $\triangle ACU$.
8. Draw any obtuse $\triangle OBT$. Use the SSS method to construct a triangle congruent to $\triangle OBT$.
9. Repeat Exercise 7, but use the SAS method.
10. Repeat Exercise 8, but use the ASA method.

On your paper, draw two angles roughly like those shown. Then for Exercises 11–14 construct an angle having the indicated measure.

11. $x + y$ 12. $x - y$ 13. $\frac{1}{2}x$ 14. $180 - 2y$

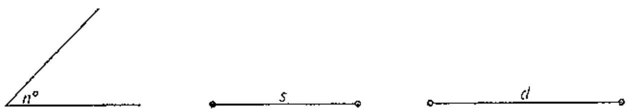
B

15. a. Draw any acute triangle. Bisect each of the three angles.
    b. Draw any obtuse triangle. Bisect each of the three angles.
    c. What do you notice about the points of intersection of the bisectors in parts (a) and (b)?
16. Construct a six-pointed star using the following procedure.
    1. Draw a ray, $\overrightarrow{AB}$. On $\overrightarrow{AB}$ mark off, in order, points C and D such that $AB = BC = CD$.
    2. Construct equilateral $\triangle ADG$.
    3. On $\overline{AG}$ mark off points E and F so that both AE and EF equal AB.
    4. On $\overline{GD}$ mark off points H and I so that both GH and HI equal AB.
    5. To complete the star, draw the three lines $\overleftrightarrow{FH}$, $\overleftrightarrow{EB}$, and $\overleftrightarrow{CI}$.

Construct an angle having the indicated measure.

17. 120 18. 150 19. 165 20. 45
21. Draw any $\triangle ABC$. Construct $\triangle DEF$ so that $\triangle DEF \sim \triangle ABC$ and $DE = 2AB$.
22. Construct a $\triangle RST$ such that $RS:ST:TR = 4:6:7$.

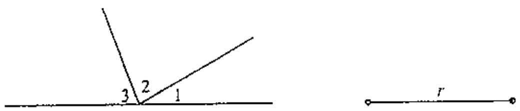
On your paper draw figures roughly like those shown. Use them in constructing the figures described in Exercises 23–25.



23. An isosceles triangle with a vertex angle of $n°$ and legs of length d
24. An isosceles triangle with a vertex angle of $n°$ and base of length s

C 25. A parallelogram with an $n°$ angle, longer side of length s, and longer diagonal of length d

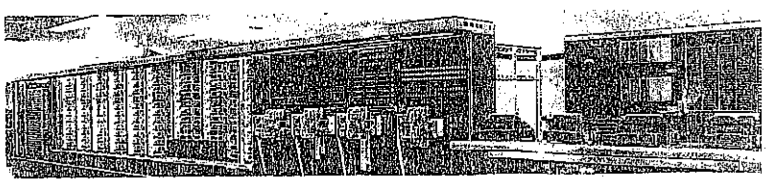
★ 26. On your paper draw figures roughly like the ones shown. Then construct a triangle whose three angles are congruent to $\angle 1$, $\angle 2$, and $\angle 3$, and whose circumscribed circle has radius r.



Biographical Note *Grace Hopper*

In 1944 the Mark I, the first working computing machine, started operations at Harvard. It could do three additions per second; calculations that took six months by hand could now be done in a day. Today, computers are one *billion* times as fast, partly because software (programming) has become more efficient, but mostly because of advances in hardware (electronics) such as the development of integrated circuits and silicon chips.

Rear Adm. Grace Hopper, U.S. Navy (Ret.) worked on that first computing machine and many others since. After getting her Ph.D. in mathematics in 1934 from Yale and teaching for several years, Hopper joined the Navy in 1943 and was assigned to Harvard as a programmer of the Mark I. In 1957, her work on making programming faster and easier resulted in her language called Flowmatic, based on the novel idea of using English words in a computer language. The first machine-independent language, COBOL, was announced in 1960 and was based on her language. She continues today to promote computers and learning, saying computers are the "first tool to assist man's brain instead of his arm."



5

Mixed Review Exercises

Complete.

1. A median of a triangle is a segment from a vertex to the __?__ of the opposite side.
2. A quadrilateral with both pairs of opposite angles congruent is a __?__.
3. A parallelogram with congruent diagonals is a __?__.
4. A parallelogram with perpendicular diagonals is a __?__.
5. If a side of a square has length 5 cm, then a diagonal of the square has length __?__ cm.
6. The measure of each interior angle of a regular pentagon is __?__.

10-2 *Perpendiculars and Parallels*

The next three constructions are based on a theorem and postulate from earlier chapters. The theorem and postulate are repeated here for your use.

(1) If a point is equidistant from the endpoints of a segment, then the point lies on the perpendicular bisector of the segment.

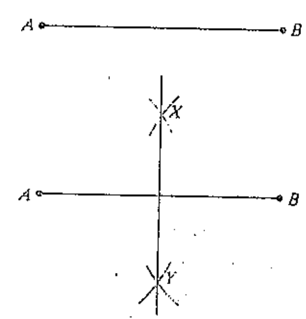
(2) Through any two points there is exactly one line.

Construction 4

Given a segment, construct the perpendicular bisector of the segment.

Given: $\overline{AB}$

Construct: The perpendicular bisector of $\overline{AB}$



Procedure:

1. Using any radius greater than $\frac{1}{2}AB$, draw four arcs of equal radii, two with center A and two with center B. Label the points of intersections of these arcs X and Y.
2. Draw $\overleftrightarrow{XY}$.

$\overleftrightarrow{XY}$ is the perpendicular bisector of $\overline{AB}$.

Justification: Points X and Y are equidistant from A and B. Thus $\overleftrightarrow{XY}$ is the perpendicular bisector of $\overline{AB}$.

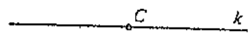
6

Construction 5

Given a point on a line, construct the perpendicular to the line at the given point.

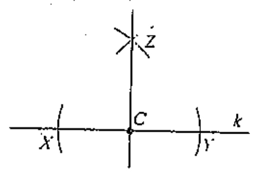
Given: Point C on line k

Construct: The perpendicular to k at C



Procedure:

1. Using C as center and any radius, draw arcs intersecting k at X and Y.
2. Using X as center and a radius greater than CX, draw an arc. Using Y as center and the same radius, draw an arc intersecting the arc with center X at a point Z.
3. Draw $\overleftrightarrow{CZ}$.



$\overleftrightarrow{CZ}$ is perpendicular to k at C.

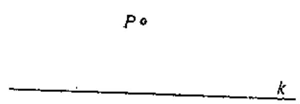
Justification: You constructed points X and Y so that C is equidistant from X and Y. Then you constructed point Z so that Z is equidistant from X and Y. Thus $\overleftrightarrow{CZ}$ is the perpendicular bisector of $\overline{XY}$, and $\overleftrightarrow{CZ} \perp k$ at C.

Construction 6

Given a point outside a line, construct the perpendicular to the line from the given point.

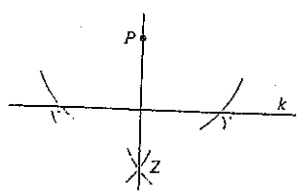
Given: Point P outside line k

Construct: The perpendicular to k from P



Procedure:

1. Using P as center, draw two arcs of equal radii that intersect k at points X and Y.
2. Using X and Y as centers and a suitable radius, draw arcs that intersect at a point Z.
3. Draw $\overleftrightarrow{PZ}$.

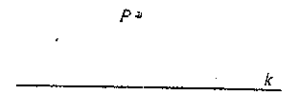


$\overleftrightarrow{PZ}$ is perpendicular to k.

Justification: Both P and Z are equidistant from X and Y. Thus $\overleftrightarrow{PZ}$ is the perpendicular bisector of $\overline{XY}$, and $\overleftrightarrow{PZ} \perp k$.

Construction 7

Given a point outside a line, construct the parallel to the given line through the given point.

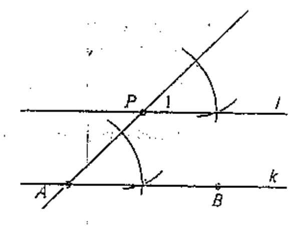


Given: Point P outside line k

Construct: The line through P parallel to k

Procedure:

1. Let A and B be two points on line k. Draw $\overrightarrow{PA}$.
2. At P, construct $\angle 1$ so that $\angle 1$ and $\angle PAB$ are congruent corresponding angles. Let l be the line containing the ray you just constructed.



l is the line through P parallel to k.

Justification: If two lines are cut by a transversal and corresponding angles are congruent, then the lines are parallel. (Postulate 11)

Classroom Exercises

1. Suggest an alternative procedure for Construction 7 that uses Constructions 5 and 6.

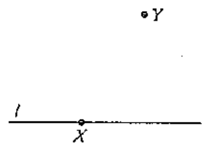
Describe how you would construct each of the following.

2. The midpoint of $\overline{BC}$
3. The median of $\triangle ABC$ that contains vertex B
4. The altitude of $\triangle ABC$ that contains vertex B
5. The altitude of $\triangle ABC$ that contains vertex A
6. The perpendicular to $\overline{BC}$ at C
7. A square whose sides each have length AC
8. A square whose perimeter equals AC
9. A right triangle with hypotenuse and one leg equal to AC and BC, respectively
10. A triangle whose sides are in the ratio $1:2:\sqrt{5}$

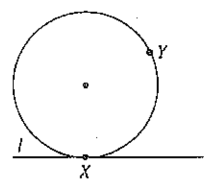
Exercises 11–13 will analyze the following problem.

Given: Line l; points X and Y

Construct: A circle through Y and tangent to l at X



If the problem had been solved, we would have a diagram something like the one shown.

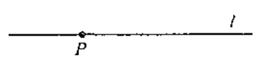


11. Where does the center of the circle lie with respect to line l and point X?
12. Where does the center of the circle lie with respect to $\overline{XY}$?
13. Explain how to carry out the construction of the circle.

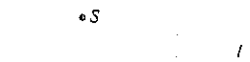
Written Exercises

Draw a figure roughly like the one shown, but larger. Do the indicated construction clearly enough so that your method can be understood easily.

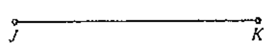
A 1. The perpendicular to l at P



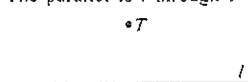
2. The perpendicular to l from S



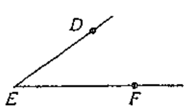
3. The perpendicular bisector of $\overline{JK}$



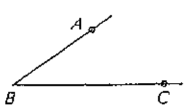
4. The parallel to l through T



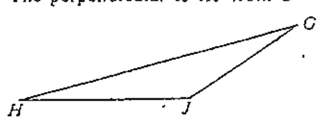
5. The parallel to $\overleftrightarrow{ED}$ through F



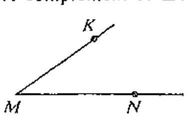
6. The perpendicular to $\overrightarrow{BA}$ at A



7. The perpendicular to $\overleftrightarrow{HJ}$ from G



8. A complement of $\angle KMN$



Construct an angle with the indicated measure.

9. 45
10. 135
11. $22\frac{1}{2}$
12. 105
13. Draw a segment $\overline{AB}$. Construct a segment $\overline{XY}$ whose length equals $\frac{3}{4}AB$.

B 14. a. Draw an acute triangle. Construct the perpendicular bisector of each side.
   b. Do the perpendicular bisectors intersect in one point?
   c. Repeat parts (a) and (b) using an obtuse triangle.
15. a. Draw an acute triangle. Construct the three altitudes.
   b. Do the lines that contain the altitudes intersect in one point?
   c. Repeat parts (a) and (b) using an obtuse triangle.
16. a. Draw a very large acute triangle. Construct the three medians.
   b. Do the lines that contain the medians intersect in one point?
   c. Repeat parts (a) and (b) using an obtuse triangle.

On your paper draw figures roughly like those shown. Use them in constructing the figures described in Exercises 17–24.

17. A parallelogram with an $n°$ angle and sides of lengths a and b
18. A rectangle with sides of lengths a and b
19. A square with perimeter $2a$
20. A rhombus with diagonals of lengths a and b
21. A square with diagonals of length b
22. A segment of length $\sqrt{a^2 + b^2}$
23. A square with diagonals of length $b\sqrt{2}$
24. A right triangle with hypotenuse of length a and one leg of length b

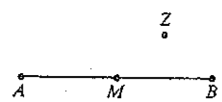
C 25. Draw a segment and let its length be s. Construct a segment whose length is $s\sqrt{3}$.
26. Draw a diagram roughly like the one shown. Without laying your straightedge across any part of the lake, construct more of $\overrightarrow{RS}$.

27. Draw three noncollinear points R, S, and T. Construct a triangle whose sides have R, S, and T as midpoints. (*Hint:* How is $\overline{RT}$ related to the side of the triangle that has S as its midpoint?)

28. Draw a segment and let its length be 1.
   a. Construct a segment of length $\sqrt{5}$.
   b. Construct a segment of length $\frac{1}{2} + \frac{\sqrt{5}}{2}$, or $\frac{1 + \sqrt{5}}{2}$.
   c. Construct a *golden rectangle* (as discussed on page 253) whose sides are in the ratio $1 : \frac{1 + \sqrt{5}}{2}$.

Challenge

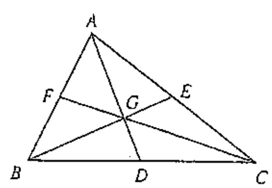
Given $\overline{AB}$, its midpoint M, and a point Z outside $\overline{AB}$, use only a straightedge (and *no* compass) to construct a line through Z parallel to $\overleftrightarrow{AB}$. (*Hint:* Use Ceva's Theorem, Exercise 33, page 273.)



Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

1. Draw any $\triangle ABC$. Draw the bisectors of the angles of the triangle. They should intersect in one point. Draw a perpendicular segment from this point to each of the sides. Measure the length of each perpendicular segment. What do you notice?
2. a. Draw any acute $\triangle ABC$. Draw the perpendicular bisector of each side of the triangle. They should intersect in one point. Measure the distance from this point of intersection to each of the vertices of the triangle. What do you notice?
   b. Repeat using an obtuse triangle and a right triangle. Is the same result true for these triangles as well?
   c. In a right triangle, the perpendicular bisectors of the sides intersect in what point?
3. Draw any $\triangle ABC$. Draw the three medians. They should intersect in one point, as shown in the diagram at the right. Find the ratios $\frac{AG}{AD}$, $\frac{BG}{BE}$, and $\frac{CG}{CF}$. What do you notice?



10-3 *Concurrent Lines*

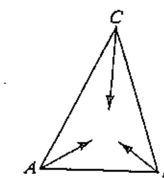
When two or more lines intersect in one point, the lines are said to be **concurrent.** For example, as you saw in Exercise 15, page 378, the bisectors of the angles of a triangle are concurrent.

Theorem 10-1

The bisectors of the angles of a triangle intersect in a point that is equidistant from the three sides of the triangle.

Given: $\triangle ABC$; the bisectors of $\angle A$, $\angle B$, and $\angle C$

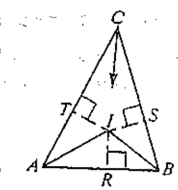
Prove: The angle bisectors intersect in a point; that point is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.



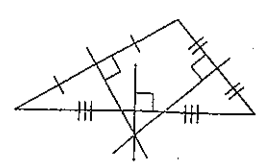
Proof:

The bisectors of $\angle A$ and $\angle B$ intersect at some point I. We will show that point I also lies on the bisector of $\angle C$ and that I is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.

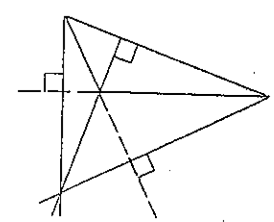
Draw perpendiculars from I intersecting $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ at R, S, and T, respectively. Since any point on the bisector of an angle is equidistant from the sides of the angle (Theorem 4-7, page 154), $IT = IR$ and $IR = IS$. Thus $IT = IS$. Since any point equidistant from the sides of an angle is on the bisector of the angle (Theorem 4-8, page 154), I is on the bisector of $\angle C$. Since $IR = IS = IT$, point I is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.



In Exercises 14–16, page 384, you discovered three other sets of concurrent lines related to triangles: the perpendicular bisectors of the sides, the lines containing the altitudes, and the medians. As you can see in the diagrams below, concurrent lines may intersect in a point outside the triangle. The intersection point may also lie on the triangle (see Classroom Exercise 4, page 388).



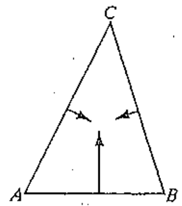
Perpendicular bisectors



Lines containing altitudes

Theorem 10-2

The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the three vertices of the triangle.



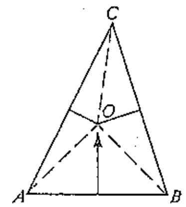
Given: $\triangle ABC$; the $\perp$ bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$

Prove: The $\perp$ bisectors intersect in a point; that point is equidistant from A, B, and C.

Proof:

The perpendicular bisectors of $\overline{AC}$ and $\overline{BC}$ intersect at some point O. We will show that point O lies on the perpendicular bisector of $\overline{AB}$ and is equidistant from A, B, and C.

Draw $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$. Since any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment (Theorem 4-5, page 153), $OA = OC$ and $OC = OB$. Thus $OA = OB$. Since any point equidistant from the endpoints of a segment lies on the perpendicular bisector of the segment (Theorem 4-6, page 153), O is on the perpendicular bisector of $\overline{AB}$. Since $OA = OB = OC$, point O is equidistant from A, B, and C.



The following theorems will be proved in Chapter 13.

Theorem 10-3

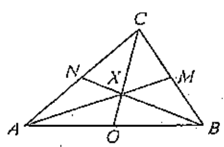
The lines that contain the altitudes of a triangle intersect in a point.

Theorem 10-4

The medians of a triangle intersect in a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

According to Theorem 10-4, if $\overline{AM}$, $\overline{BN}$, and $\overline{CO}$ are medians of $\triangle ABC$, then:

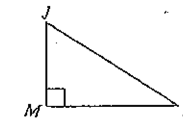
$$AX = \tfrac{2}{3}AM$$
$$XN = \tfrac{1}{3}BN$$
$$CX:XO:CO = 2:1:3$$



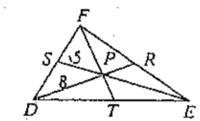
The points of intersection described in the theorems in this section are sometimes called the *incenter* (point where the angle bisectors meet), *circumcenter* (point where the perpendicular bisectors meet), *orthocenter* (point where the altitudes meet), and *centroid* (point where the medians meet).

Classroom Exercises

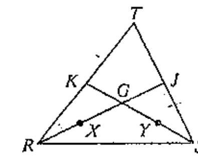
1. Draw, if possible, a triangle in which the perpendicular bisectors of the sides intersect in a point with the location described.
   a. A point inside the triangle
   b. A point outside the triangle
   c. A point on the triangle
2. Repeat Exercise 1, but work with angle bisectors.
3. Is there some kind of triangle such that the perpendicular bisector of each side is also an angle bisector, a median, and an altitude?
4. $\triangle JAM$ is a right triangle.
   a. Is $\overline{JM}$ an altitude of $\triangle JAM$?
   b. Name another altitude shown.
   c. In what point do the three altitudes of $\triangle JAM$ meet?
   d. Where do the perpendicular bisectors of the sides of $\triangle JAM$ meet?
   e. Does your answer to (d) agree with Theorem 10-2?



5. The medians of $\triangle DEF$ are shown. Find the lengths indicated.
   a. EP = _?_
   b. PR = _?_
   c. If $FT = 9$, then PT = _?_ and FP = _?_.



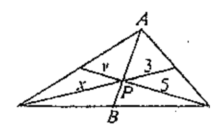
6. Given: $\overline{RJ}$ and $\overline{SK}$ are medians of $\triangle RST$;
   X and Y are the midpoints of $\overline{RG}$ and $\overline{SG}$.
   a. How are $\overline{XY}$ and $\overline{RS}$ related? Why?
   b. How are $\overline{KJ}$ and $\overline{RS}$ related? Why?
   c. How are $\overline{KJ}$ and $\overline{XY}$ related? Why?
   d. What special kind of quadrilateral is $XYJK$? Why?
   e. Why does $XG = GJ$?
   f. Explain why $RG = \tfrac{2}{3}RJ$.



Written Exercises

A 1. Draw a triangle such that the lines containing the three altitudes intersect in a point with the location described.
   a. A point inside the triangle
   b. A point outside the triangle
   c. A point on the triangle

Exercises 2–5 refer to the diagram in which the medians of a triangle are shown.

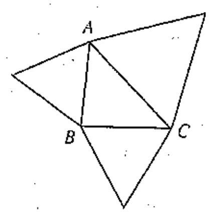


2. Find the values of x and y.
3. If $AB = 6$, then BP = _?_ and AP = _?_.
4. If $AB = 7$, then BP = _?_ and AP = _?_.
5. If $PB = 1.9$, then AP = _?_ and AB = _?_.

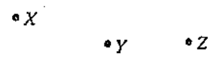
6. Use a ruler and a protractor to draw a regular pentagon. Then construct the perpendicular bisectors of the five sides.

7. Draw a regular pentagon as in Exercise 6. Construct the angle bisectors.

8. Draw any large $\triangle ABC$ and construct equilateral triangles on each of the sides as shown.
   a. In each of the three equilateral triangles, construct any two medians and find their point of intersection.
   b. Draw the three segments connecting these three points of intersection.
   c. What appears to be true about the triangle you drew in part (b)?

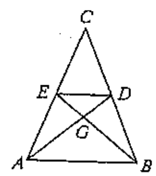


B 9. Three towns, located as shown, plan to build one recreation center to serve all three towns. They decide that the fair thing to do is to build the hall equidistant from the three towns. Comment about the wisdom of the plan.



10. See Exercise 9. Locate three towns so that it isn't possible to find a spot equidistant from the three towns.

11. In the figure, $\overline{AD}$ and $\overline{BE}$ are congruent medians of $\triangle ABC$.
   a. Explain why $GD = GE$.
   b. $GA =$ __?__
   c. Name three angles congruent to $\angle GAB$.

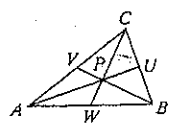


$\overline{AU}$, $\overline{BV}$, and $\overline{CW}$ are the medians of $\triangle ABC$.

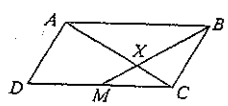
12. If $AP = x^2$ and $PU = 2x$, then $x =$ __?__.

13. If $BP = y^2 + 1$ and $PV = y + 2$, then $y =$ __?__ or $y =$ __?__.

14. If $CW = 2z^2 - 5z - 12$ and $CP = z^2 - 15$, then $z =$ __?__ and $PW =$ __?__.

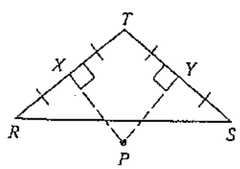


15. $ABCD$ is a parallelogram with M the midpoint of $\overline{CD}$. If $\overline{BM}$ intersects $\overline{AC}$ at X, prove that $CX = \frac{1}{3}AC$. (*Hint:* Draw $\overline{BD}$.)



16. Prove that if two of the medians of a triangle are congruent, then the triangle is isosceles.

C 17. In the plane figure, point P is equidistant from R, S, and T. Describe the location of the following points in the plane.
   a. Points farther from both R and S than from T
   b. Points closer to both R and S than to T



15

Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

1. Inscribe a circle D inside a $\triangle ABC$. Draw $\overline{DA}$, $\overline{DB}$, and $\overline{DC}$. Compare the measures of $\angle ABD$ and $\angle ABC$, $\angle ACD$ and $\angle ACB$, $\angle BAD$ and $\angle BAC$. What do you notice? What type of lines intersect at the center of a circle inscribed in a triangle?

2. Circumscribe a circle D about a $\triangle ABC$. Draw perpendicular segments from D to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, intersecting the sides at E, F, and G, respectively. Compare the lengths of $\overline{AE}$ and $\overline{AB}$, $\overline{BF}$ and $\overline{BC}$, and $\overline{CG}$ and $\overline{CA}$. What do you notice? What type of lines intersect at the center of a circle circumscribed about a triangle?

More Constructions

Objectives

1. Perform seven additional basic constructions.
2. Use the basic constructions in original construction exercises.

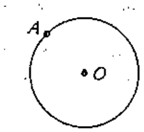
10-4 *Circles*

Construction 8

Given a point on a circle, construct the tangent to the circle at the given point.

Given: Point A on $\odot O$

Construct: The tangent to $\odot O$ at A

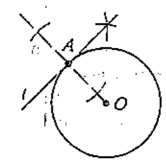


Procedure:

1. Draw $\overrightarrow{OA}$.
2. Construct the line perpendicular to $\overrightarrow{OA}$ at A. Call it t.

Line t is tangent to $\odot O$ at A.

Justification: Because t is perpendicular to radius $\overline{OA}$ at A, t is tangent to $\odot O$.



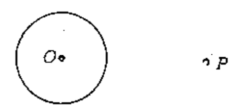
16

Construction 9

Given a point outside a circle, construct a tangent to the circle from the given point.

Given: Point P outside $\odot O$

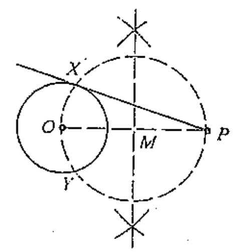
Construct: A tangent to $\odot O$ from P



Procedure:

1. Draw $\overline{OP}$.
2. Find the midpoint M of $\overline{OP}$ by constructing the perpendicular bisector of $\overline{OP}$.
3. Using M as center and MP as radius, draw a circle that intersects $\odot O$ in a point X.
4. Draw $\overrightarrow{PX}$.

$\overrightarrow{PX}$ is tangent to $\odot O$ from P. $\overrightarrow{PY}$, not drawn, is the other tangent from P.



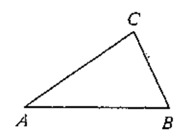
Justification: If you draw $\overline{OX}$, $\angle OXP$ is inscribed in a semicircle. Then $\angle OXP$ is a right angle and $\overrightarrow{PX} \perp \overline{OX}$. Because $\overrightarrow{PX}$ is perpendicular to radius $\overline{OX}$ at its outer endpoint, $\overrightarrow{PX}$ is tangent to $\odot O$.

Construction 10

Given a triangle, circumscribe a circle about the triangle.

Given: $\triangle ABC$

Construct: A circle passing through A, B, and C

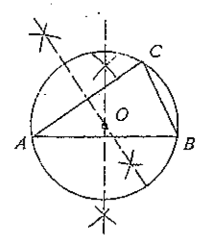


Procedure:

1. Construct the perpendicular bisectors of any two sides of $\triangle ABC$. Label the point of intersection O.
2. Using O as center and OA as radius, draw a circle.

Circle O passes through A, B, and C.

Justification: See Theorem 10-2 on page 387.

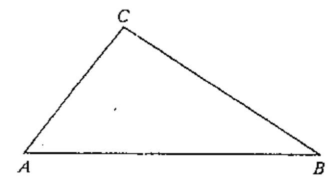


Construction 11

Given a triangle, inscribe a circle in the triangle.

Given: $\triangle ABC$

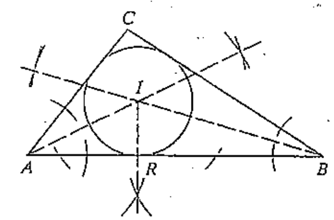
Construct: A circle tangent to $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$



Procedure:

1. Construct the bisectors of $\angle A$ and $\angle B$. Label the point of intersection I.
2. Construct a perpendicular from I to $\overline{AB}$, intersecting $\overline{AB}$ at a point R.
3. Using I as center and IR as radius, draw a circle.

Circle I is tangent to $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.

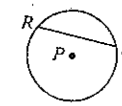


Justification: See Theorem 10-1 on page 386.

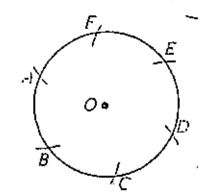
Classroom Exercises

1. Explain how to find the midpoint of $\widehat{AB}$.
2. Explain how to construct the center of the circle containing points A, B, and C.

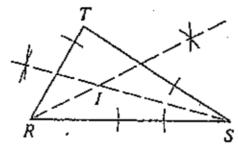
3. Explain how to find the line described.
   a. Parallel to $\overline{RS}$ and passing through P
   b. Parallel to $\overline{RS}$ and tangent to $\odot P$



4. Here you see a common method for using just one compass setting for drawing a circle and dividing the circle into six congruent arcs. Explain how the method works.
5. Suppose a circle is given. Explain how you can use the method of Exercise 4 to inscribe an equilateral triangle in the circle.
6. Suppose the construction of Exercise 4 has been carried out. Explain how you can then inscribe a regular twelve-sided polygon in the circle.



7. A student intends to inscribe a circle in $\triangle RST$. The center I has been found as shown. How should the student find the radius needed?



Written Exercises

In Exercises 1 and 2 draw a diagram similar to the one shown, but larger.

A 1. Construct a tangent at A.

2. Construct two tangents from P.

3. Draw a large acute triangle. Construct the circumscribed circle.

4. Construct a large right triangle. Construct the circumscribed circle.

5. Draw a large obtuse triangle. Construct the circumscribed circle.

6. Draw a large acute triangle. Construct the inscribed circle.

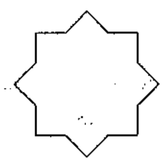
7. Construct a large right triangle. Construct the inscribed circle.

8. Draw a large obtuse triangle. Construct the inscribed circle.

B 9. Draw a circle. Inscribe an equilateral triangle in the circle.

10. Draw a circle. Inscribe a square in the circle.

11. a. Draw a circle. Inscribe a regular octagon in the circle.
    b. How would you use your construction in part (a) to create an eight-pointed star as shown at the right?



Ex. 11(b)

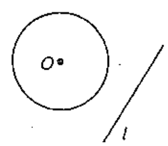
12. Draw a circle. Circumscribe a square about the circle.

13. Construct a square. Circumscribe a circle about the square.

14. Construct a square. Inscribe a circle in the square.

15. Draw a circle. Circumscribe an equilateral triangle about the circle.

In each of Exercises 16 and 17 begin with a diagram roughly like the one shown, but larger.



16. Construct a line that is parallel to line l and tangent to $\odot O$.

17. Construct a line that is perpendicular to line l and tangent to $\odot O$.

C 18. Construct three congruent circles, each tangent to the other two circles. Then construct an equilateral triangle, each side of which is tangent to two of the circles.

In Exercises 19–21 begin with two circles P and Q such that $\odot P$ and $\odot Q$ do not intersect and Q is not inside $\odot P$. Let the radii of $\odot P$ and $\odot Q$ be p and q respectively, with $p > q$.

19. Construct a circle, with radius equal to PQ, that is tangent to $\odot P$ and $\odot Q$.

20. Construct a common external tangent to $\odot P$ and $\odot Q$. One method is suggested below.
    1. Draw a circle with center P and radius $p - q$.
    2. Construct a tangent to this circle from Q, and call the point of tangency Z.
    3. Draw $\overrightarrow{PZ}$. $\overrightarrow{PZ}$ intersects $\odot P$ in a point X.
    4. With center X and radius ZQ, draw an arc that intersects $\odot Q$ in a point Y.
    5. Draw $\overleftrightarrow{XY}$.

    As a justification for this construction, you could begin by drawing $\overline{QY}$. Then show that $XZQY$ is a rectangle. The rest of the justification is easy.

21. Construct a common internal tangent to $\odot P$ and $\odot Q$. (*Hint:* Draw a circle with center P and radius $p + q$.)

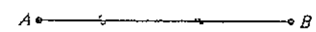
10-5 *Special Segments*

Construction 12

Given a segment, divide the segment into a given number of congruent parts. (3 shown)

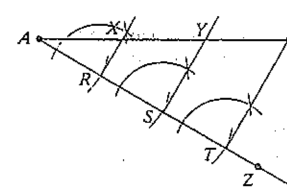
Given: $\overline{AB}$

Construct: Points X and Y on $\overline{AB}$ so that $AX = XY = YB$



Procedure:

1. Choose any point Z not on $\overleftrightarrow{AB}$. Draw $\overrightarrow{AZ}$.
2. Using any radius, start with A as center and mark off R, S, and T so that $AR = RS = ST$.
3. Draw $\overline{TB}$.
4. At R and S construct lines parallel to $\overline{TB}$, intersecting $\overline{AB}$ in X and Y.



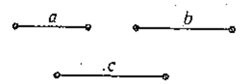
$\overline{AX}$, $\overline{XY}$, and $\overline{YB}$ are congruent parts of $\overline{AB}$.

Justification: Since the parallel lines you constructed cut off congruent segments on transversal $\overleftrightarrow{AZ}$, they cut off congruent segments on transversal $\overleftrightarrow{AB}$. (It may help you to think of the parallel to $\overline{TB}$ through A.)

Construction 13

Given three segments, construct a fourth segment so that the four segments are in proportion.

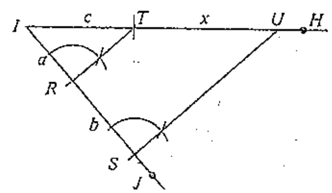
Given: Segments with lengths a, b, and c



Construct: A segment of length x such that $\frac{a}{b} = \frac{c}{x}$

Procedure:

1. Draw an $\angle HIJ$.
2. On $\overrightarrow{IJ}$, mark off $IR = a$ and $RS = b$.
3. On $\overrightarrow{IH}$, mark off $IT = c$.
4. Draw $\overline{RT}$.
5. At S, construct a parallel to $\overline{RT}$, intersecting $\overrightarrow{IH}$ in a point U.



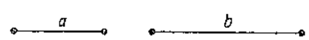
$\overline{TU}$ has length x such that $\frac{a}{b} = \frac{c}{x}$.

Justification: Because $\overline{RT}$ is parallel to side $\overline{SU}$ of $\triangle SIU$, $\overline{RT}$ divides the other two sides of the triangle proportionally. Therefore, $\frac{a}{b} = \frac{c}{x}$.

Construction 14

Given two segments, construct their geometric mean.

Given: Segments with lengths a and b

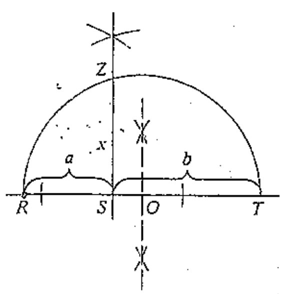


Construct: A segment of length x such that $\frac{a}{x} = \frac{x}{b}$

(or $x = \sqrt{ab}$)

Procedure:

1. Draw a line and mark off $RS = a$ and $ST = b$.
2. Locate the midpoint O of $\overline{RT}$ by constructing the perpendicular bisector of $\overline{RT}$.
3. Using O as center draw a semicircle with a radius equal to OR.
4. At S, construct a perpendicular to $\overline{RT}$. The perpendicular intersects the semicircle at a point Z.



ZS, or x, is the geometric mean between a and b.

Justification: $\widehat{RZT}$ is a semicircle. If you draw $\overline{RZ}$ and $\overline{ZT}$, then $\triangle RZT$ is a right triangle. Since $\overline{ZS}$ is the altitude to the hypotenuse of rt. $\triangle RZT$, $\frac{a}{x} = \frac{x}{b}$.

Classroom Exercises

1. Given a segment, tell how to construct an equilateral triangle whose perimeter equals the length of the given segment.

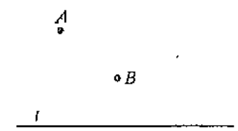
Draw three segments and label their lengths a, b, and c.

2. Construct a segment of length x such that $\frac{c}{a} = \frac{b}{x}$.
3. Describe how to construct a segment of length x such that $x = \sqrt{2ab}$.
4. Describe how to construct a segment of length x such that $x = \sqrt{5ab}$.
5. Describe how to construct a segment of length x such that $x = \sqrt{4ab}$.

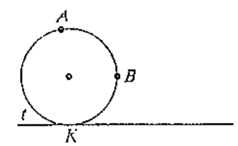
Exercises 6–11 will analyze the following problem.

Given: Line t; points A and B

Construct: A circle through A and B and tangent to t

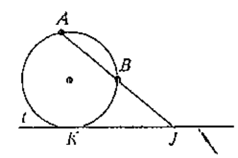


If the problem had been solved, we would have a diagram something like the one shown.



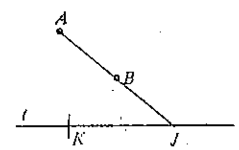
6. Where does the center of the circle lie with respect to $\overline{AB}$?
7. Where does the center of the circle lie with respect to line t and K, the point of tangency?

Note that we don't have point K located in the given diagram. Hunting for ideas, we draw $\overleftrightarrow{AB}$. We now have a point J, which we can locate in the given diagram.



8. State an equation that relates JK to JA and JB.
9. Rewrite your equation in the form $\frac{?}{JK} = \frac{JK}{?}$.
10. What construction can we use to get the length JK?

In a *separate* diagram we can mark off the lengths JA and JB on some line l and then use Construction 14 to find x such that $\frac{JA}{x} = \frac{x}{JB}$. Once we have x, which equals JK, we return to the given diagram and draw an arc to locate K.



11. Explain how to complete the construction of the circle.

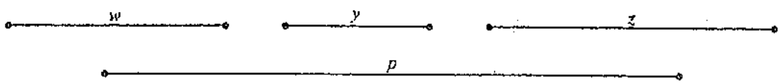
Written Exercises

In each of Exercises 1–4 begin by drawing $\overline{AB}$ roughly 15 cm long.

A 1. Divide $\overline{AB}$ into three congruent segments.

2. a. Use Construction 12 to divide $\overline{AB}$ into four congruent segments.
   b. Use Construction 4 to divide $\overline{AB}$ into four congruent segments.
3. a. Use Construction 12 to divide $\overline{AB}$ into five congruent segments.
   b. Can Construction 4 be used to divide $\overline{AB}$ into five congruent segments?
   c. Divide $\overline{AB}$ into two segments that have the ratio 2:3.
4. Divide $\overline{AB}$ into two segments that have the ratio 3:4.

On your paper draw four segments roughly as long as those shown below. Use your segments in Exercises 5–14. In each exercise construct a segment that has length x satisfying the given condition.



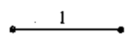
5. $\frac{y}{w} = \frac{z}{x}$ 6. $\frac{w}{x} = \frac{x}{y}$ 7. $x = \sqrt{yp}$ 8. $3x = w + 2y$

B 9. $zx = wy$ (*Hint:* First write a proportion that is equivalent to the given equation and has x as the last term.)

10. $x = \frac{yp}{z}$ 11. $x = \frac{1}{3}\sqrt{yp}$ 12. $x = \sqrt{3wz}$ 13. $x = \sqrt{6yz}$

14. Construct $\overline{AB}$, with $AB = p$. Divide $\overline{AB}$ into two parts that have the ratio $w:y$.

15. Draw a segment like the one shown and let its length be 1. Use the segment to construct a segment of length $\sqrt{15}$.

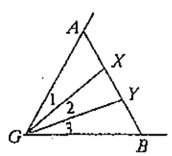


16. a. If $x = a\sqrt{n}$, then x is the geometric mean between a and __?__.
    b. Draw a segment about 3 cm long. Call its length a. Use your results from part (a) to construct a segment of length $a\sqrt{n}$ for n = 2, 3, and 4.

C 17. Draw $\overline{CD}$ about 20 cm long. Construct a triangle whose perimeter is equal to CD and whose sides are in the ratio 2:2:3.

★ 18. To trisect a general angle G, a student tried this procedure:
1. Mark off $\overline{GA}$ congruent to $\overline{GB}$.
2. Draw $\overline{AB}$.
3. Divide $\overline{AB}$ into three congruent parts using Construction 12.
4. Draw $\overline{GX}$ and $\overline{GY}$.

Show that the student did not trisect $\angle G$. (*Hint:* Show that $GA > GY$. Then use an indirect proof to show that $m\angle 2 \neq m\angle 1$.)



Self-Test 2

1. Draw a large $\odot O$. Choose a point A that is outside $\odot O$. Construct the two tangents to $\odot O$ from point A.
2. Draw a very large obtuse triangle. Construct the inscribed circle.
3. Draw a segment about half as long as the width of your paper. Then divide the segment by construction into two segments whose lengths have the ratio 2:1.
4. Draw a large $\triangle ABC$. Then construct $\overline{DE}$ such that $\frac{AB}{BC} = \frac{AC}{DE}$.
5. Use $\triangle ABC$ drawn in Exercise 4 to construct a segment, $\overline{PQ}$, whose length is the geometric mean of AB and AC.
6. You are given $\odot S$ and diameter $\overline{FG}$. To construct parallel tangents to $\odot S$, you could construct a line that is __?__ to $\overline{FG}$ at __?__ and a line that is __?__ to $\overline{FG}$ at __?__.
7. You are given $\triangle TRI$. Describe the steps you would use to circumscribe a circle about $\triangle TRI$.

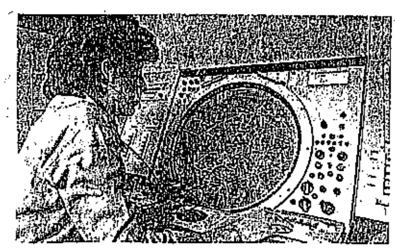
Locus

Objectives

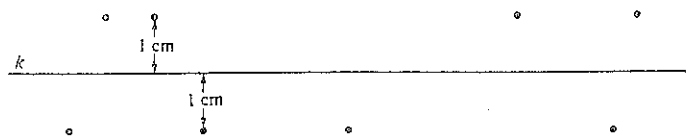
1. Describe the locus that satisfies a given condition.
2. Describe the locus that satisfies more than one given condition.
3. Apply the concept of locus in the solution of construction exercises.

10-6 *The Meaning of Locus*

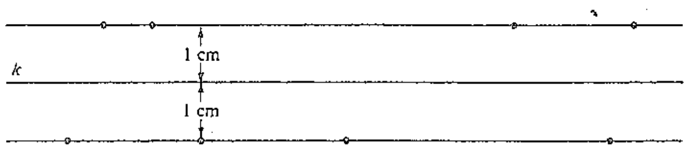
A radar system is used to determine the position, or *locus*, of airplanes relative to an airport. In geometry **locus** means a figure that is the set of all points, and only those points, that satisfy one or more conditions.



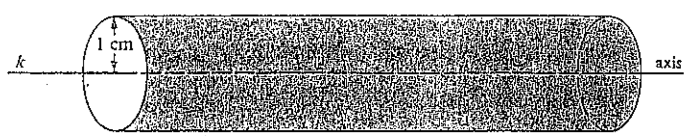
Suppose we have a line k in a plane and wish to picture the locus of points in the plane that are 1 cm from k. Several points are shown in the first diagram below.



All the points satisfying the given conditions are indicated in the next diagram. You see that the required locus is a pair of lines parallel to, and 1 cm from, k.



Suppose we wish to picture the locus of points 1 cm from k without requiring the points to be *in a plane*. The problem changes. Now you need to consider all the points in space that are 1 cm from line k. The required locus is a cylindrical surface with axis k and a 1 cm radius, as shown below. Of course, the surface will extend in both directions without end, just as line k does.

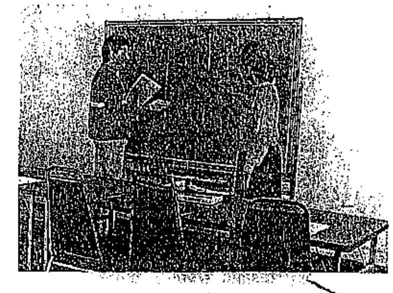


When you are solving a locus problem, always think in terms of three dimensions unless the statement of the problem restricts the locus to a plane.

Classroom Exercises

1. Draw a point A on the chalkboard.
   a. Draw several points on the chalkboard that are 20 cm from A.
   b. Draw all the points on the chalkboard that are 20 cm from A.
   c. Complete: The locus of all points on the chalkboard that are 20 cm from point A is _?_.
   d. Remove the restriction that the points must lie in the plane of the chalkboard. Now describe the locus.

2. Draw two parallel lines k and l.
   a. Draw several points that are in the plane containing k and l and are equidistant from k and l.
   b. Draw all the points that are in the plane containing k and l and are equidistant from k and l.
   c. Describe the locus of points that are in the plane of two parallel lines and equidistant from them.
   d. Remove the restriction that the points must lie in the plane of the two lines. Now describe the locus.
3. Draw an angle.
   a. Draw several points in the plane of the angle that are equidistant from the sides of the angle.
   b. Draw all the points in the plane of the angle that are equidistant from the sides of the angle.
   c. Describe the locus of points in the plane of a given angle that are equidistant from the sides of the angle.
4. What is the locus of points in your classroom that are equidistant from the ceiling and floor?
5. What is the locus of points in your classroom that are 1 m from the floor?
6. Choose a point P on the floor of the classroom.
   a. What is the locus of points, on the floor, that are 1 m from P?
   b. What is the locus of points, in the room, that are 1 m from P?
7. What is the locus of points in your classroom that are equidistant from the ceiling and floor and are also equidistant from two opposite side walls?



8. What is the locus of points in your classroom that are equidistant from the front and back walls and are also equidistant from the two side walls?
9. Describe the locus of points on a football field that are equidistant from the two goal lines.
10. Draw a circle with radius 6 cm. Use the following definition of *distance from a circle*: A point P is x cm from a circle if there is a point of the circle that is x cm from P but there is no point of the circle that is less than x cm from P.
    a. Draw all the points in the plane of the circle that are 2 cm from the circle.
    b. Complete: Given a circle with a 6 cm radius, the locus of all points in the plane of the circle and 2 cm from the circle is _?_.
    c. Remove the restriction that the points must lie in the plane of the circle. Now describe the locus.
11. Make up a locus problem for which the locus contains exactly one point.
12. Make up a locus problem for which the locus doesn't contain any points.

Written Exercises

Exercises 1–4 deal with figures in a plane. Draw a diagram showing the locus. Then write a description of the locus.

1. Given two points A and B, what is the locus of points equidistant from A and B?
2. Given a point P, what is the locus of points 2 cm from P?
3. Given a line h, what is the locus of points 2 cm from h?
4. Given $\odot O$, what is the locus of the midpoints of all radii of $\odot O$?

In Exercises 5–8 begin each exercise with a square $ABCD$ that has sides 4 cm long. Draw a diagram showing the locus of points on or inside the square that satisfy the given conditions. Then write a description of the locus.

5. Equidistant from $\overline{AB}$ and $\overline{CD}$
6. Equidistant from points B and D
7. Equidistant from $\overline{AB}$ and $\overline{BC}$
8. Equidistant from all four sides

Exercises 9–12 deal with figures in space.

9. Given two parallel planes, what is the locus of points equidistant from the two planes?
10. Given a plane, what is the locus of points 5 cm from the plane?
11. Given point E, what is the locus of points 3 cm from E?
12. Given points C and D, what is the locus of points equidistant from C and D?

Exercises 13–17 deal with figures in a plane. (*Note*: If a point in a segment or in an arc is not included in the locus, indicate the point by an open dot.)

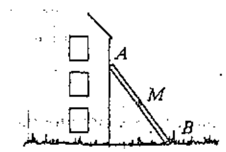
B 13. a. Draw an angle HEX. Construct the locus of points equidistant from the sides of $\angle HEX$.
b. Draw two intersecting lines j and k. Construct the locus of points equidistant from j and k.

14. Draw a segment $\overline{DE}$ and a line n. Construct the locus of points whose distance from n is DE.
15. Draw a segment $\overline{AB}$. Construct the locus of points P such that $\angle APB$ is a right angle.
16. Draw a segment $\overline{CD}$. Construct the locus of points Q such that $\triangle CQD$ is isosceles with base $\overline{CD}$.
17. Draw a segment $\overline{EF}$. Construct the locus of points G such that $\triangle EFG$ is isosceles with leg $\overline{EF}$.

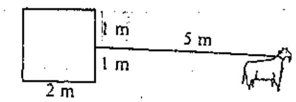
Exercises 18–20 deal with figures in space.

18. Given a sphere, what is the locus of the midpoints of the radii of the sphere?
19. Given a square, what is the locus of points equidistant from the sides?
20. Given a scalene triangle, what is the locus of points equidistant from the vertices?

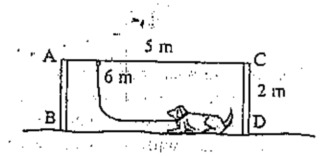
C 21. A ladder leans against a house. As A moves up or down on the wall, B moves along the ground. What path is followed by midpoint M? (*Hint:* Experiment with a meter stick, a wall, and the floor.)



22. Given a segment $\overline{CD}$, what is the locus in space of points P such that $m\angle CPD = 90$?
23. A goat is tied to a square shed as shown. Using the scale 1:100, carefully draw a diagram that shows the region over which the goat can graze.



24. A tight wire $\overline{AC}$ is stretched between the tops of two vertical posts $\overline{AB}$ and $\overline{CD}$ that are 5 m apart and 2 m high. A ring, at one end of a 6 m leash, can slide along $\overline{AC}$. A dog is tied to the other end of the leash. Draw a diagram that shows the region over which the leashed dog can roam. Use the scale 1:100.

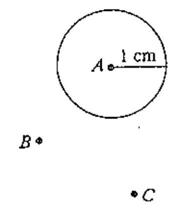


10-7 *Locus Problems*

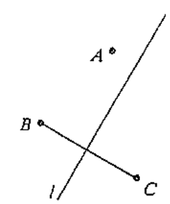
The plural of *locus* is *loci*. The following problem involves intersections of loci.

Suppose you are given three noncollinear points, A, B, and C. In the plane of A, B, and C, what is the locus of points that are 1 cm from A and are, at the same time, equidistant from B and C?

You can analyze one part of the problem at a time.

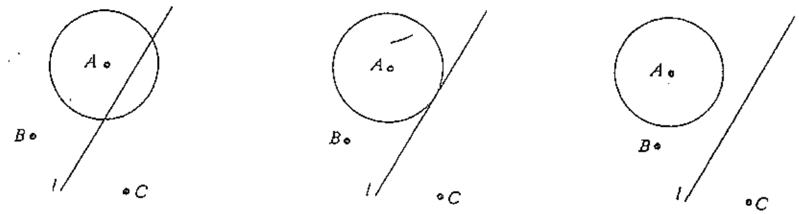


The locus of points 1 cm from A is $\odot A$ with radius 1 cm.



The locus of points equidistant from B and C is l, the perpendicular bisector of $\overline{BC}$.

The locus of points satisfying *both* conditions given on the previous page must lie on both circle A and line l. There are three possibilities, depending on the positions of A, B, and C, as shown below.



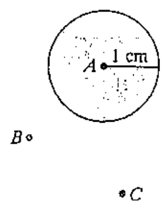
All three can be described in one sentence:

The locus is two points, one point, or no points, depending on the intersection of the circle with center A and radius 1 cm and the line that is the perpendicular bisector of $\overline{BC}$.

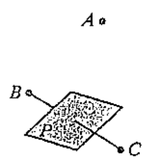
The example that follows deals with the corresponding problem in three dimensions.

Example Given three noncollinear points A, B, and C, what is the locus of points 1 cm from A and equidistant from B and C?

Solution



The first locus is sphere A with radius 1 cm.



The second locus is plane P, the perpendicular bisector of $\overline{BC}$.

Possibilities:

The plane might cut the sphere in a circle.
The plane might be tangent to the sphere.
The plane might not have any points in common with the sphere.

Thus, the locus is a circle, one point, or no points, depending on the intersection of the sphere with center A and radius 1 cm and the plane which is the perpendicular bisector of $\overline{BC}$.

Classroom Exercises

Exercises 1–4 refer to coplanar figures. Describe the possible intersections of the figures named.

1. A line and a circle
2. Two circles
3. Two parallel lines and a circle
4. Two perpendicular lines and a circle

5. Consider the following problem: In a plane, what is the locus of points that are equidistant from the sides of $\angle A$ and are equidistant from two points B and C?
   a. The locus of points equidistant from the sides of $\angle A$ is _?_.
   b. The locus of points equidistant from B and C is _?_.
   c. Draw diagrams to show three possibilities with regard to points that satisfy both conditions (a) and (b).
   d. Describe the locus.

Exercises 6–9 refer to figures in space. Describe the possible intersections of the figures named.

6. A line and a plane
7. A line and a sphere
8. Two spheres
9. A plane and a sphere

10. Let C be the point in the center of your classroom (*not* the center of the floor). Describe the locus of points in the room that satisfy the given conditions.
    a. 3 m from C
    b. 3 m from C and equidistant from the ceiling and the floor
    c. 3 m from C and 1 m from either the ceiling or the floor

Written Exercises

Exercises 1 and 2 refer to plane figures.

A
1. Draw a new $\odot O$ for each part. Then place two points A and B outside $\odot O$ so that the locus of points on $\odot O$ and equidistant from A and B is:
   a. 2 points
   b. 0 points
   c. 1 point
2. Draw two parallel lines m and n. Then place two points R and S so that the locus of points equidistant from m and n and also equidistant from R and S is:
   a. 1 point
   b. 1 line
   c. 0 points

Exercises 3 and 4 refer to plane figures.

3. Consider the following problem: Given two points D and E, what is the locus of points 1 cm from D and 2 cm from E?
   a. The locus of points 1 cm from D is _?_.
   b. The locus of points 2 cm from E is _?_.
   c. Draw diagrams to show three possibilities with regard to points that satisfy both conditions (a) and (b).
   d. Give a one-sentence solution to the problem.
4. Consider the following problem: Given a point A and a line k, what is the locus of points 3 cm from A and 1 cm from k?
   a. The locus of points 3 cm from A is _?_.
   b. The locus of points 1 cm from k is _?_.
   c. Draw diagrams to show five possibilities with regard to points that satisfy both conditions (a) and (b).
   d. Give a one-sentence solution to the problem.

Exercises 5–10 refer to plane figures. Draw a diagram of the locus. Then write a description of the locus.

5. Point P lies on line l. What is the locus of points on l and 3 cm from P?
6. Point Q lies on line l. What is the locus of points 5 cm from Q and 3 cm from l?
7. Points A and B are 3 cm apart. What is the locus of points 2 cm from both A and B?
8. Lines j and k intersect in point P. What is the locus of points equidistant from j and k, and 2 cm from P?
9. Given $\angle A$, what is the locus of points equidistant from the sides of $\angle A$ and 2 cm from vertex A?
10. Given $\triangle RST$, what is the locus of points equidistant from $\overline{RS}$ and $\overline{RT}$ and also equidistant from R and S?

In Exercises 11–14 draw diagrams to show the possibilities with regard to points in a plane.

11. Given points C and D, what is the locus of points 2 cm from C and 3 cm from D?
12. Given point E and line k, what is the locus of points 3 cm from E and 2 cm from k?
13. Given a point A and two parallel lines j and k, what is the locus of points 30 cm from A and equidistant from j and k?
14. Given four points P, Q, R, and S, what is the locus of points that are equidistant from P and Q and equidistant from R and S?

Exercises 15–19 refer to figures in space. In each exercise tell what the locus is. You need not draw the locus or describe it precisely.

Example Given two parallel planes and a point A, what is the locus of points equidistant from the planes and 3 cm from A?

Solution The locus is a circle, a point, or no points.

15. Given plane Z and point B outside Z, what is the locus of points in Z that are 3 cm from B?
16. Given plane Y and point P outside Y, what is the locus of points 2 cm from P and 2 cm from Y?
17. Given $\overleftrightarrow{AB} \perp$ plane Q, what is the locus of points 2 cm from $\overleftrightarrow{AB}$ and 2 cm from Q?
18. Given square $ABCD$, what is the locus of points equidistant from the vertices of the square?
19. Given point A in plane Z, what is the locus of points 5 cm from A and d cm from Z? (More than 1 possibility)
20. Given three points, each 2 cm from the other two, draw a diagram to show the locus of points that are in the plane of the given points and are not more than 2 cm away from any of them.
21. Points R, S, T, and W are not coplanar and no three of them are collinear.
   a. The locus of points equidistant from R and S is _?_.
   b. The locus of points equidistant from R and T is _?_.
   c. The loci found in parts (a) and (b) intersect in a _?_, and all points in this _?_ are equidistant from points R, S, and T.
   d. The locus of points equidistant from R and W is _?_.
   e. The intersection of the figures found in (c) and (d) is a _?_. This _?_ is equidistant from the four given points.

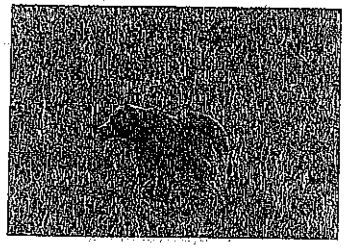
C 22. Can you locate four points J, K, L, and M so that the locus of points equidistant from J, K, L, and M is named below? If the answer is *yes*, describe the location of the points J, K, L, and M.
   a. a point
   b. a line
   c. a plane
   d. no points

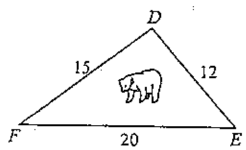
23. Assume that the Earth is a sphere. How many points are there on the Earth's surface that are equidistant from
   a. Houston and Toronto?
   b. Houston, Toronto, and Los Angeles?
   c. Houston, Toronto, Los Angeles, and Mexico City?

24. A mini-radio transmitter has been secured to a bear. Rangers D, E, and F are studying the bear's movements. Rangers D and E can receive the bear's beep at distances up to 10 km, ranger F at distances up to 15 km.

Draw a diagram showing where the bear might be at these times:

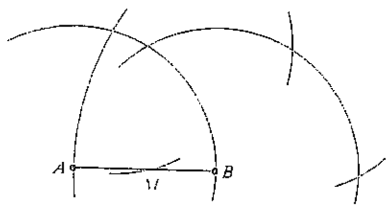
a. When all three rangers can receive the signal
b. When ranger F suddenly detects the signal after a period of time during which only rangers D and E could receive the signal
c. When ranger D is off duty, and ranger F begins to detect the signal just as ranger E loses it





Challenge

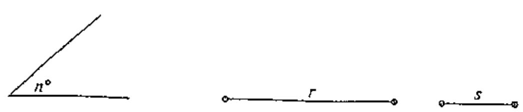
Given $\overline{AB}$, it is possible to construct the midpoint M of $\overline{AB}$ using only a compass (and *no* straightedge). Study the diagram until you understand the procedure. Then draw $\overline{AB}$, about 10 cm long, construct its midpoint M as shown, and prove that M is the midpoint.



10-8 *Locus and Construction*

Sometimes the solution to a construction problem depends on finding a point that satisfies more than one condition. To locate the point, you may have to begin by constructing a locus of points satisfying *one* of the conditions.

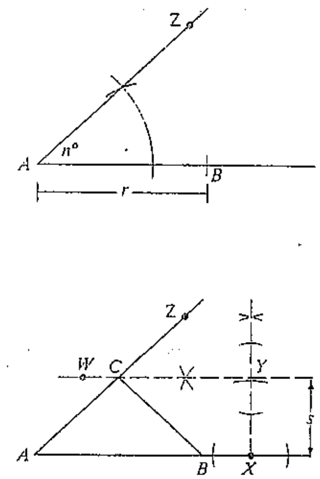
Example Given the angle and the segments shown, construct $\triangle ABC$ with $m\angle A = n$, $AB = r$, and the altitude to $\overleftrightarrow{AB}$ having length s.



Solution It is easy to construct $\angle A$ and side $\overline{AB}$. Point C must satisfy two conditions: C must lie on $\overrightarrow{AZ}$, and C must be s units from $\overleftrightarrow{AB}$. The locus of points s units from $\overleftrightarrow{AB}$ is a pair of parallel lines. Only the upper parallel will intersect $\overrightarrow{AZ}$. We construct that parallel to $\overleftrightarrow{AB}$ as follows:

1. Construct the perpendicular to $\overleftrightarrow{AB}$ at any convenient point X.
2. Mark off s units on the perpendicular to locate point Y.
3. Construct the perpendicular to $\overrightarrow{XY}$ at Y. Call it $\overleftrightarrow{YW}$.

Note that all points on $\overleftrightarrow{YW}$ are s units from $\overleftrightarrow{AB}$. Thus the intersection of $\overleftrightarrow{YW}$ and $\overrightarrow{AZ}$ is the desired point C. To complete the solution, we simply draw $\overline{CB}$.

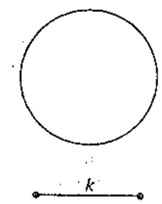


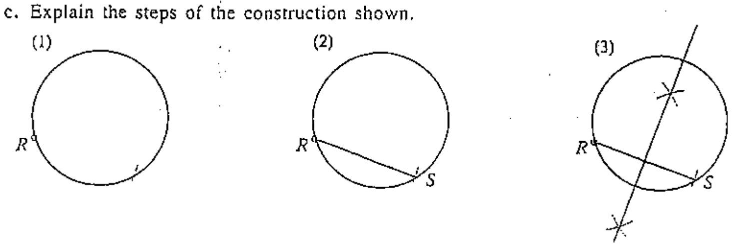
Classroom Exercises

1. The purpose of this exercise is to analyze the following construction problem:

   Given a circle and a segment with length k, inscribe in the circle an isosceles triangle RST with base $\overline{RS}$ k units long.

   a. Suppose R has been chosen. Where must S lie so that RS equals k? (In other words, what is the locus of points k units from R?)
   b. Now suppose $\overline{RS}$ has been drawn. Where must T lie so that $RT = ST$? (In other words, what is the locus of points equidistant from R and S?)
   c. Explain the steps of the construction shown.





   d. Explain two different ways to finish the construction.

33

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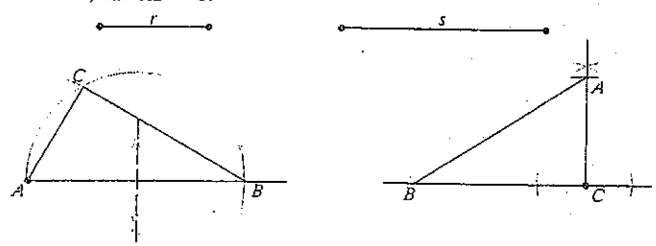
2. Two different solutions, both correct, are shown for the following construction problem. Analyze the diagrams and explain the solutions.

Given segments with lengths r and s, construct $\triangle ABC$ with $m\angle C = 90$, $AC = r$, and $AB = s$.



First solution   Second solution

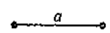
Written Exercises

Exercises 1–4 refer to plane figures.

A 1. Draw any $\overline{AB}$ and a segment with length h. Use the following steps to construct the locus of points P such that for every $\triangle APB$ the altitude from P to $\overleftrightarrow{AB}$ would equal h.
   a. Construct a perpendicular to $\overline{AB}$.
   b. Construct two lines parallel to $\overline{AB}$, h units from $\overline{AB}$.

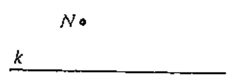
2. Begin each part of this exercise by drawing any $\overline{CD}$. Then construct the locus of points P that meet the given condition.
   a. $\angle CDP$ is a right angle.
   b. $\angle CPD$ is a right angle. (*Hint:* See Classroom Exercise 2.)

On your paper draw a segment roughly as long as the one shown. Use it in Exercises 3 and 4.

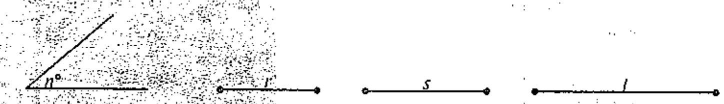


3. Draw an angle XYZ. Construct a circle, with radius a, that is tangent to the sides of $\angle XYZ$. (*Hint:* The center of the circle will be a units from the sides of $\angle XYZ$.)

4. Draw a figure roughly like the one shown. Then construct a circle, with radius a, that passes through N and is tangent to line k. (*Hint:* Construct the locus of points that would, as centers, be the correct distance from k. Also construct the locus of points that would, as centers, be the correct distance from N.)



On your paper draw an angle and three segments roughly like those shown. Use them in Exercises 5–19. You may find it helpful to begin with a sketch.



5. Construct $\overline{AB}$ so that $AB = t$. Then construct the locus of all points C so that in $\triangle ABC$ the altitude from C has length r.

6. Construct $\overline{AB}$ so that $AB = t$. Then construct the locus of all points C so that in $\triangle ABC$ the median from C has length s.

B 7. Construct isosceles $\triangle ABC$ so that $AB = AC = t$ and so that the altitude from A has length s.

8. Construct an isosceles trapezoid $ABCD$ with $\overline{AB}$ the shorter base, with $AB = AD = BC = t$, and with an altitude of length r.

9. Construct $\triangle ABC$ so that $AB = t$, $AC = s$, and the median to $\overline{AB}$ has length r.

10. Construct $\triangle ABC$ so that $m\angle A = m\angle B = n$, and the altitude to $\overline{AB}$ has length s.

11. Construct $\triangle ABC$ so that $m\angle C = 90$, $m\angle A = n$, and the altitude to $\overline{AB}$ has length s.

12. Construct $\triangle ABC$ so that $AB = s$, $AC = t$, and the altitude to $\overline{AB}$ has length r.

13. Construct $\triangle ABC$ so that $AB = t$, and the median to $\overline{AB}$ and the altitude to $\overline{AB}$ have lengths s and r, respectively.

14. Construct a right triangle such that the altitude to the hypotenuse and the median to the hypotenuse have lengths r and s, respectively.

15. Construct both an acute isosceles triangle and an obtuse isosceles triangle such that each leg has length s and each altitude to a leg has length r.

C 16. Construct a square whose sides each have length $4s$. A segment of length $3s$ moves so that its endpoints are always on the sides of the square. Construct the locus of the midpoint of the moving segment.

17. Construct a right triangle such that the bisector of the right angle divides the hypotenuse into segments whose lengths are r and s.

18. Construct an isosceles right triangle such that the radius of the inscribed circle is r.

19. Construct $\overline{AB}$ so that $AB = t$. Then construct the locus of points P such that $m\angle APB = n$.

(12) Division of a given segment into any number of congruent parts, page 396

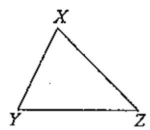
(13) A segment of length x such that $\frac{a}{b} = \frac{c}{x}$ when segments of lengths a, b, and c are given, page 397

(14) A segment whose length is the geometric mean between the lengths of two given segments, page 397

3. Every triangle has these concurrency properties:
   (1) The bisectors of the angles intersect in a point that is equidistant from the three sides of the triangle.
   (2) The perpendicular bisectors of the sides intersect in a point that is equidistant from the three vertices of the triangle.
   (3) The lines that contain the altitudes intersect in a point.
   (4) The medians intersect in a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.
4. A locus is the set of all points, and only those points, that satisfy one or more conditions.
5. A locus that satisfies more than one condition is found by considering all possible intersections of the loci for the separate conditions.

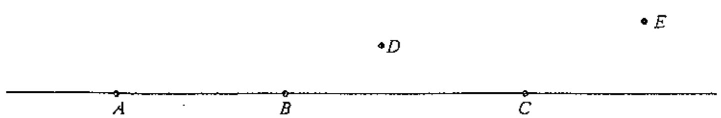
Chapter Review

In Exercises 1–3 draw a diagram that is similar to, but larger than, the one shown. Then do the constructions.



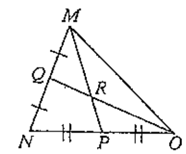
1. Draw any line m. On m construct $\overline{ST}$ such that $ST = 3XY$. 10–1
2. Construct an angle with measure equal to $m\angle X + m\angle Z$.
3. Bisect $\angle Y$.

Use a diagram like the one below for Exercises 4–7.



4. Construct the perpendicular bisector of $\overline{AB}$. 10–2
5. Construct the perpendicular to $\overleftrightarrow{AC}$ at C.
6. Construct the perpendicular to $\overleftrightarrow{AC}$ from D.
7. Construct the parallel to $\overleftrightarrow{AC}$ through E.

8. The __?__ of a triangle intersect in a point that is equidistant from the vertices of the triangle. 10–3
9. The __?__ of a triangle intersect in a point that is equidistant from the sides of the triangle.
10. If $MR = 12$, then $MP =$ __?__.
11. $QR:RO =$ __?__ (numerical answer)

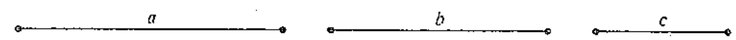


Exs. 10, 11

Draw a large ⊙O. Label a point F on ⊙O and a point G outside ⊙O.

12. Construct the tangent to ⊙O at F. 10–4
13. Construct a tangent to ⊙O from G.
14. Draw a large acute triangle. Find, by construction, the center of the circle that could be inscribed in the triangle.
15. Draw a large obtuse triangle. Construct a circle that circumscribes the triangle.

Draw segments about as long as those shown below. In each exercise, construct a segment with the required length t.



16. $t^2 = bc$ 17. $at = bc$ 18. $t = \frac{1}{3}(a + b)$ 10–5
19. Given two parallel lines l and m, what is the locus of points in their plane and equidistant from them? 10–6
20. Given two points A and B, what is the locus of points, in space, equidistant from A and B?
21. What is the locus of points in space equidistant from two parallel planes?
22. What is the locus of points in space that are equidistant from the vertices of equilateral $\triangle HJK$?
23. Points P and Q are 6 cm apart. What is the locus of points in a plane that are equidistant from P and Q and are 8 cm from P? Sketch the locus. 10–7
24. Point R is on line l. What is the locus in space of points that are 8 cm from l and 8 cm from R?
25. What is the locus of points in space that are 1 m from plane Q and 2 m from point Z not in Q? (There is more than one possibility.)

Use the segments with lengths a, b, and c that you drew for Exercises 16–18.

26. Construct an isosceles right triangle with hypotenuse of length a. 10–8
27. Construct a $\triangle RST$ with $RS = a$, $RT = c$, and the median to $\overline{RS}$ of length b.