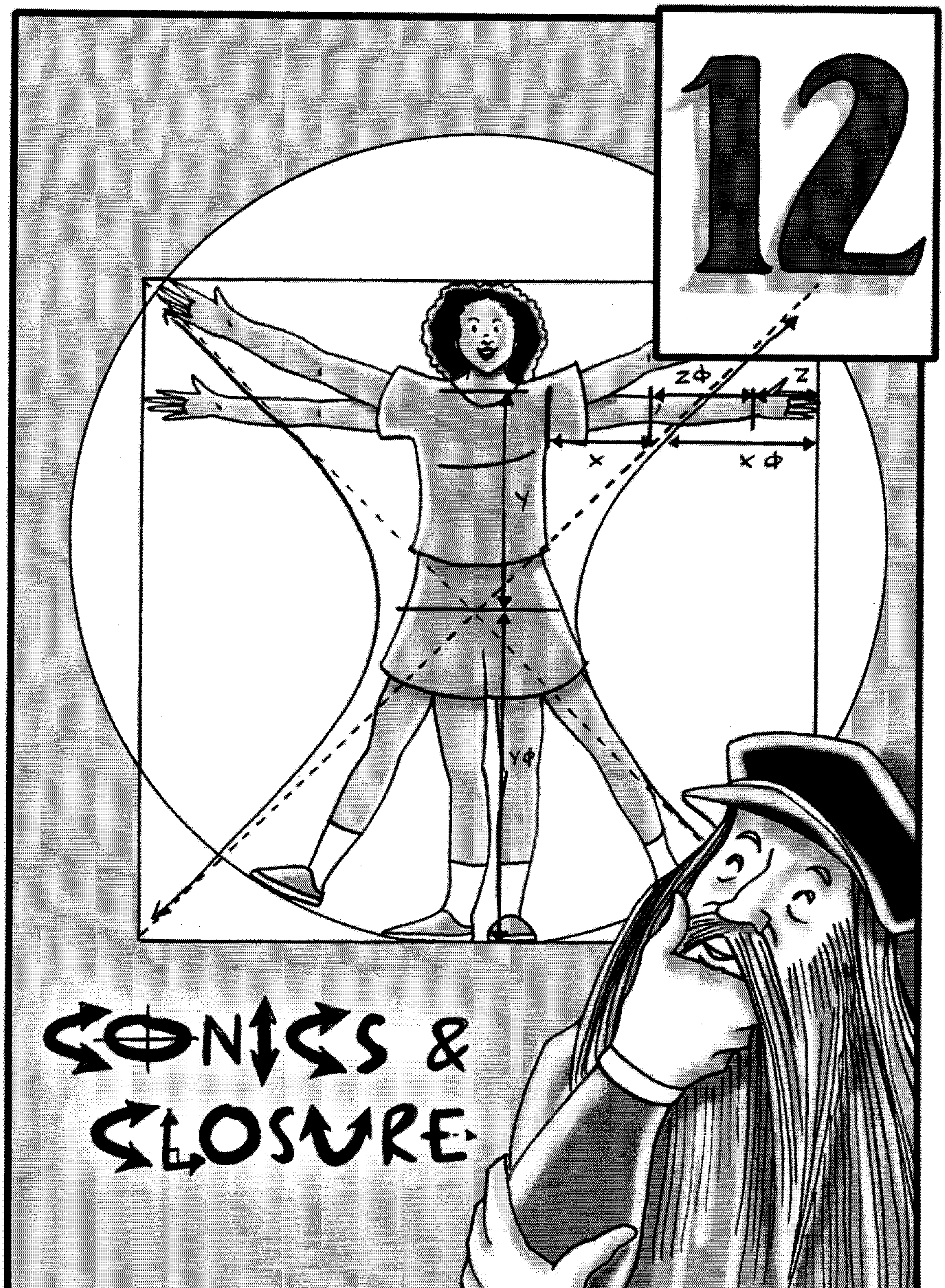
12
CONICS &
CLOSURE
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CHAPTER 12 Conics and Closure

As this course draws to a close, it is appropriate to reflect on what you have learned so far and to look for connections between topics in both algebra and geometry.

For example, in Section 12.1, you will extend your understanding of the cross-sections of a cone, called "conic sections." You will learn about the geometric properties of conic sections and will discover how to represent them algebraically.

Then, in Section 12.2, four activities offer a chance for you to apply your geometric **tools** in new ways. You will find new connections between familiar geometric ideas and learn even more special properties about familiar shapes.

Guiding Questions

Think about these questions throughout this chapter:

What's the cross-section?

How can I draw it?

What's the relationship?

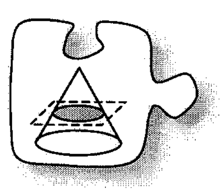
What information do I need?

Is there another way?

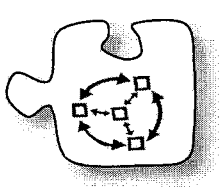
In this chapter, you will learn:

- How to identify the cross-section of a solid.
- How to represent the cross-sections of a cone (the "conic sections") algebraically.
- The geometric definitions of a circle, a parabola, an ellipse, and a hyperbola.

Chapter Outline



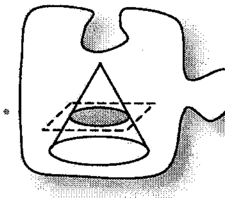
Section 12.1 By studying the different cross-sections of a cone, you will discover how geometry and algebra can each define a shape. You will learn how to construct several conic sections using tools, such as string and tracing paper, and will develop geometric definitions for a circle, parabola, ellipse, and hyperbola.

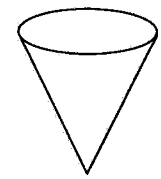


Section 12.2 As you complete the course closure activities, you will apply the mathematics you have learned throughout this course. For example, you will discover a special ratio that often occurs in nature, find new relationships that exist in basic polyhedra, learn about a shape created when the midpoints of the sides of a quadrilateral are connected, and determine where a goat should be tethered to a barn so that it has the lowest probability of eating a poisoned weed.

12.1.1 What's the cross-section?

Introduction to Conic Sections





In Section 11.1, you explored the cross-sections of several types of solids. In fact, you learned that a sphere is special because it is the only solid that has a circular cross-section no matter which direction it is sliced. But what about the cross-sections of a cone? What different cross-sections can be found? And what can be learned about these cross-sections?

In Section 12.1, you and your team will explore the various cross-sections of a cone. As you explore, look for connections with other mathematical concepts you have studied previously.

12-1. CONIC SECTIONS

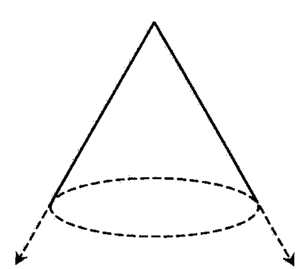
Obtain the Lesson 12.1.1 Resource Page from your teacher and construct a cone using scissors and tape. Then, with your team, explore the different cross-sections of a cone (called **conic sections**). Imagine slicing a cone as many different ways as you can. Draw and describe each cross-section on your paper. Do you know the names for any of these shapes?

12-2. **Examine** your conic sections from problem 12-1.

a. If you have not done so already, determine how you can slice the cone so that the cross-section is a ray.

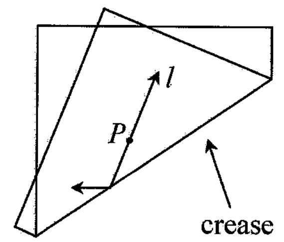
b. Can the cross-section of a cone be a single point? Explain.

c. If you have not done so already, describe the cross-section of the cone shown at right when it is sliced vertically through the top vertex.



12-3. Interestingly, one of the types of conic sections is a curve you studied in algebra: the **parabola**. What more can be learned about the geometry of a parabola?

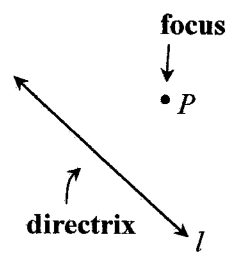
a. In Chapter 9, you constructed a parabola using tracing paper (see problem 9-78). With your team, reconstruct a parabola by carefully drawing a line (l) and a point not on the line (P) on tracing paper. Then fold the tracing paper so that line l passes through point P. Unfold the tracing paper and fold it again at a different point on line l so that line l still passes through point P. Continue this process until you have at least 20 creased lines forming a parabola.



b. **Examine** where the parabola lies in relationship with the original point P and line l. Where does the point lie? Where does the line lie? Do these relationships seem to hold for the other parabolas constructed by your teammates?

12-4. FOCUS AND DIRECTRIX OF A PARABOLA

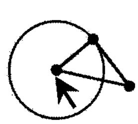
Since the point and the line help to determine the parabola, there are special names that are used to refer to them. The point is called the **focus** of the parabola, while the line is called the **directrix**.



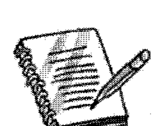
a. Together, the focus and the directrix determine the parabola. For example, can you **visualize** the parabola formed by the focus and directrix shown at right? Trace the point and line on your paper and sketch the parabola.

b. What is the relationship between the points on the parabola and its focus and directrix? Mark a point on the parabola and label it A. Fold the tracing paper so that l passes through P and the crease passes through A. Compare the distance between P and A and the distance between A and l. What do you notice? Does this work for all points on the parabola?

c. How does the distance between the focus (the point) and the directrix (the line) affect the shape of the parabola? Explore this using a dynamic geometry tool, if possible. (If a dynamic tool is not available, use tracing paper to test several different distances between the focus and directrix.) Explain the result.



12-5. Write an entry in your Learning Log describing what you learned during this lesson. Include information about the cross-sections of a cone and the geometric relationships in a parabola. What questions do you have about the other conic sections? Title this entry "Conic Sections" and include today's date.



Review & Preview

12-6. Cawker City, Kansas, claims to have the world's largest ball of twine. Started in 1953 by Frank Stoeber, this ball has been created by wrapping more than 1300 miles of twine. In fact, this giant ball has a circumference of 40 feet. Assuming the ball of twine is a sphere, find the surface area and volume of the ball of twine.

12-7. The equations below are the types of equations that you will need to be able to solve automatically in a later course. Try to solve these in 10 minutes or less. The solutions are provided after problem 12-11 for you to check your answers.

a. $2x-5=7$ b. $x^2=16$ c. $2(x-1)=6$

d. $\frac{x}{5}=6$ e. $2x^2+5=x^2+14$ f $(x-3)(x+5)=0$

12-8. **Examine** the pen or pencil that you are using right now. Imagine slicing it in different directions. On your paper, draw at least three different cross-sections of the pen or pencil.

12-9. On graph paper, graph $x^2+y^2=9$.

a. Consider the inequality $x^2+y^2\le 9$. Does the point (0, 0) make this inequality true? What is the graph of $x^2+y^2\le 9$? Explain.

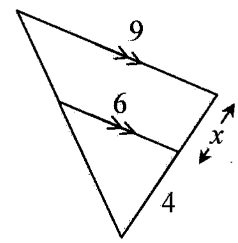
b. Now consider the inequality $x^2+y^2>9$. Does the point (0, 0) make this inequality true? What region is shaded? Describe the graph of this inequality.

12-10. Remember that the **absolute value function** finds the distance on a number line between a number and zero. For example, the absolute value of –6 (written $|-6|$) equals 6, while $|2|=2$.

On graph paper, copy and complete the table below and graph the function $y=|x|+2$.

x	–4	–3	–2	–1	0	1	2	3	4
y	6				2				

12-11. **Multiple Choice:** In the diagram at right, the value of x is:



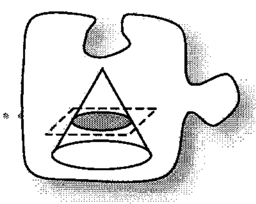
a. 1 b. 2 c. 3

d. 4 e. None of these

Solutions to problem 12-7: a: 6, b: 4 or −4, c: 4, d: 30, e: 3 or −3, f: 3 or −5

12.1.2 How can I graph it?

Graphing Parabolas Using The Focus and Directrix



In Lesson 12.1.1, you **investigated** the geometric properties of a parabola, one of the cross-sections of a cone. Today you and your team will explore other conic sections as you continue to find ways to connect geometry and algebra.

12-12. GRAPHING WITH A FOCUS AND DIRECTRIX

In the past, you have graphed conics, such as circles and parabolas, using rectangular graph paper and an equation. However, another way to graph conic sections is to use **focus-directrix paper**, that is designed with lines and concentric circles like the example shown in Figure A at right.

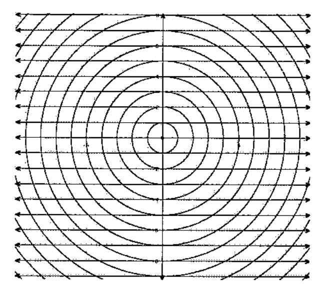


Figure A:
Focus and directrix paper

How can you graph parabolas using this paper? Obtain at least two sheets of focus-directrix paper from your teacher and follow the directions below.

a. In problem 12-4, you discovered that each point on a parabola is an equal distance from the focus and directrix. To graph a parabola, highlight the center of the concentric circles on your focus-directrix grid with a colored marker. This will be the focus of the parabola. Then highlight a line that is two units away from the focus, as shown in the Figure B at right.

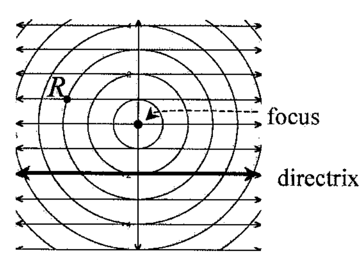


Figure B:
Point R on grid

b. **Examine** point R on the focus-directrix grid in Figure B. Notice that the circles help you count the distance between R and the focus (the center of the circles). Explain how you know that the point R is 3 units from the focus and 3 units from the directrix.

Problem continues on next page →

12-12. *Problem continued from previous page.*

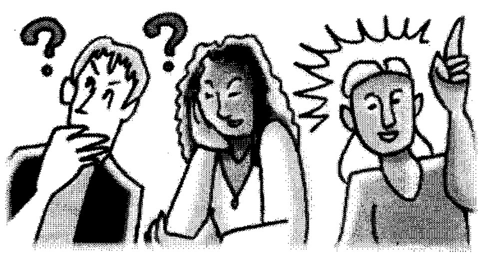
c. Use the circles and lines to plot a point that is 1 unit away from the focus and the directrix. Is there another point that is also 1 unit away from both the focus and directrix?

d. Likewise, find two points that are 2 units away from both the focus and the directrix. Continue plotting points that are equidistant from the focus and the directrix until the parabola appears. Compare your parabola with those of your teammates to double-check for accuracy.

12-13. What else can you learn about graphing parabolas using focus-directrix paper? The left column of the table below contains several **investigative** questions to explore using focus-directrix paper. For each question, first **visualize** the resulting parabola and discuss your prediction with your team. Then **test** the situation on a fresh focus-directrix grid. The right-hand column contains a suggested way to test your idea, although your team may design its own way to **investigate** the question.

Investigative Questions	**Try it out!**
a. What would happen to the parabola from problem 12-12 if the directrix were moved so that it is above the focus?	On a new focus-directrix grid, place the directrix on the line that is 2 units above the focus. Then plot the points of the resulting parabola, making sure each point is the same distance from the focus and directrix. Describe what happens.
b. What would happen to the parabola from problem 12-12 if the directrix were moved so that it is <u>farther</u> from the focus?	On a new focus-directrix grid, place the directrix so that it is 6 units away from the focus. Then plot the points of the resulting parabola, making sure each point is the same distance from the focus and directrix. Describe what happens.
c. What would happen to the parabola from problem 12-12 if the directrix were moved so that it is <u>closer</u> to the focus?	On a new focus-directrix grid, place the directrix so that it is 1 unit away from the focus. Then plot the points of the resulting parabola, making sure each point is the same distance from the focus and directrix. Describe what happens.

12-14. With your team, brainstorm your own **investigative** question you would like to explore. You may want to start the questions with "What if…" or "What happens when…" to help you get started. Share your questions with the class.



12-15. Celia asks this question: "What if the points are *closer* to the focus than the directrix?" For example, what if the distance between each point and the focus is <u>half</u> the distance between that point and the directrix? Consider this as you answer the questions below.

a. On a new sheet of focus-directrix paper, highlight the center (focus) and a line that is 6 units away from the center.

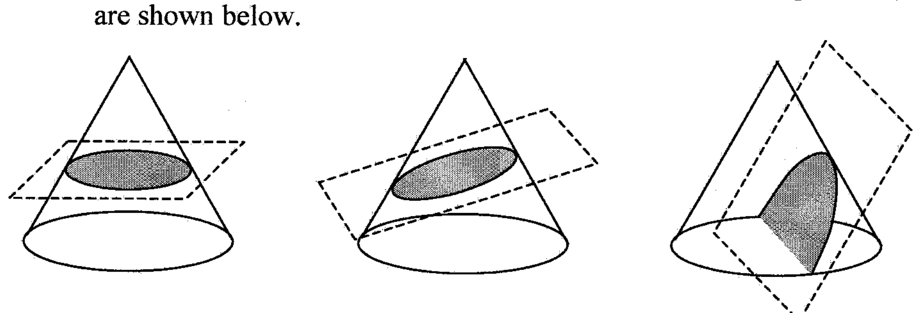
b. For every unit a point is away from the focus, it needs to be 2 units away from the directrix. Find the first point that is 2 units away from the focus and 4 units away from the directrix. Then find another point that is 1 *more* unit away from the focus and 2 *more* units away from the directrix. (This point should be a total of 3 units away from the focus and 6 units away from the directrix.) Continue this pattern until the graph is complete. What shape do you see?

MATH NOTES

METHODS AND MEANINGS

Conic Sections

The cross-sections of a cone are also called **conic sections**. The shape of the cross-section depends on the angle of the slice. Three possible cross-sections of a cone (a circle, an ellipse, and a parabola) are shown below.



Circle **Ellipse** **Parabola**

Review & Preview

12-16. A solid with volume 820 cm^3 is reduced proportionally with a linear scale factor of $\frac{1}{2}$. What is the volume of the result?

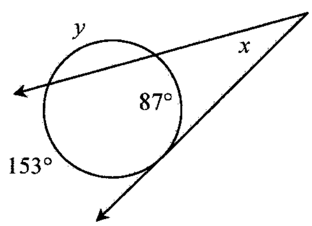
12-17. On graph paper, graph the equations below.

a. $x^2 + y^2 = 4.5^2$

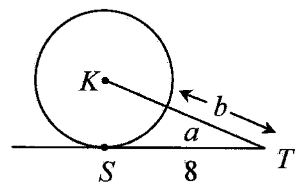
b. $x^2 + y^2 = 75$

12-18. Use the relationships in each diagram below to solve for the given variables.

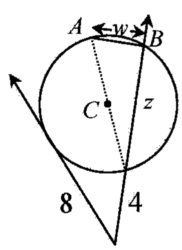
a.



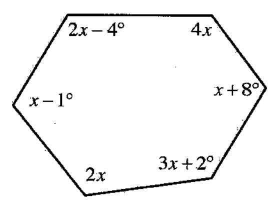
b. The area of $\odot K$ is 36π un^2.



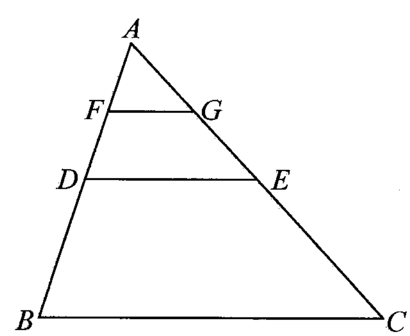
c. The diameter of $\odot C$ is 13 units. w is the length of $\overline{AB}$.



d.



12-19. In the triangle at right, $\overline{DE}$ is a midsegment of ΔABC and $\overline{FG}$ is a midsegment of ΔADE. If $DE = 7$ cm, find BC and FG.

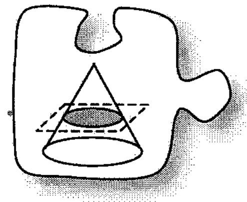


12-20. Find the volume of a cone if the circumference of the base is 28π inches and the height is 18 inches.

12-21. **Multiple Choice:** As Carol shopped for a spring picnic, she spent \$2.00 for each liter of soda and \$3.50 for each bag of chips. In all, she bought 18 items for a total of \$43.50. Assuming she only bought chips and soda, how many bags of chips did she buy?

a. 9 b. 5 c. 15 d. 3

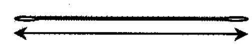
12.1.3 What shape does it make?



Circles and Ellipses

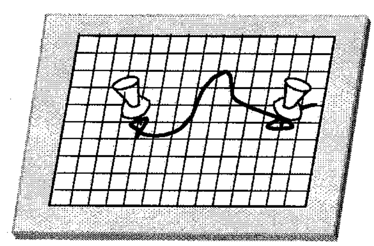
In Lesson 12.1.1, you **investigated** the geometric properties of a parabola, one of the cross-sections of a cone. Today you and your team will explore other conic sections as you look for ways to connect geometry and algebra.

12-22. What shapes can you make using string? To find out, obtain a piece of cardboard, some thumbtacks, a ruler, and a piece of string from your teacher. Then follow the directions below.



a. Attach a piece of graph paper to the cardboard. Form two loops on the string so that, when pulled apart, the ends of the loops are 10 units apart. Attach one loop to the center of the cardboard using a thumbtack. If you place your pencil in the other loop and keep the string taut (tight), what shape will you create? Explain how you know and then test your idea by drawing the shape on the graph paper.

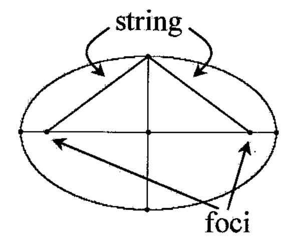
b. What if the string is attached to the cardboard at the ends of both loops? Attach both loops of the string from part (a) to the cardboard using two thumbtacks spaced 8 units apart. Predict what shape a pencil will create as it pulls the string tight in all directions.



Then test your prediction by drawing the shape on your graph paper. Describe the result. Where have you seen this shape before?

c. What happens to the shape if the thumbtacks are moved so that they are farther apart? What happens when the thumbtacks are closer together? What happens when the thumbtacks are at the same point? Explore these questions with your team and be prepared to share your conclusions with the class.

12-23. **ELLIPSE**



The shape that you created in problem 12-22 is called an **ellipse**. Each thumbtack represents a **focus** of the ellipse, so while a parabola only has one focus, an ellipse has two **foci** (plural for "focus").

Examine the ellipse that you created in problem 12-22. How wide is it (in graph paper units)? How tall? How are these lengths related to the length of the string (between the loops)?

12-24. On focus-directrix paper, graph each set of points that are described below. Use a new focus-directrix grid for each part.

a. Graph the set of points that are 6 units away from the focus.

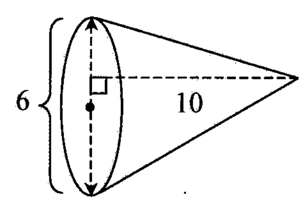
b. The directrix is 8 units away from the focus. Graph the set of set of points that are equidistant (i.e. the same distance) from the focus and the directrix.

c. The directrix is 12 units away from the focus. Graph the result if the distance between each point of the graph and the focus is half the distance between that same point on the graph and the directrix.

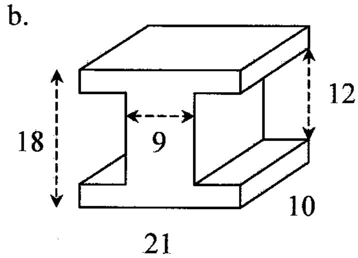
Review & Preview

12-25. Find the volume of each shape below. Assume that all corners in part (b) are right angles.

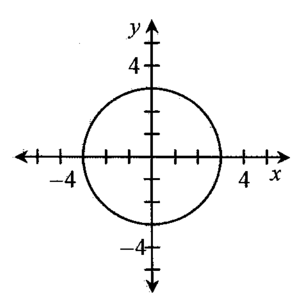
a. cone



b.



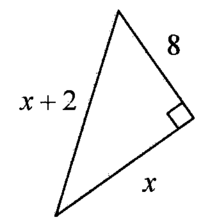
12-26. **Examine** the graph of the circle at right.



a. Find the equation of the circle.

b. On graph paper, sketch the graph of the equation $x^2 + y^2 = 49$. What is the radius?

12-27. **Examine** the diagram at right.



a. Write an equation using the geometric relationships in the diagram. Then solve your equation for x.

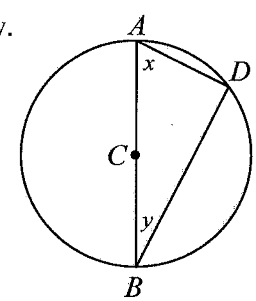
b. Find the measures of the acute angles of the triangle. What **tool** did you use?

12-28. Jinning is going to flip a coin. If the result is "heads," he wins $4. If the result is "tails," he loses $7.

a. What is his expected value per flip?

b. If he flips the coin 8 times, how much should he win or lose?

12-29. Use the diagram of $\odot C$ at right to answer the questions below.

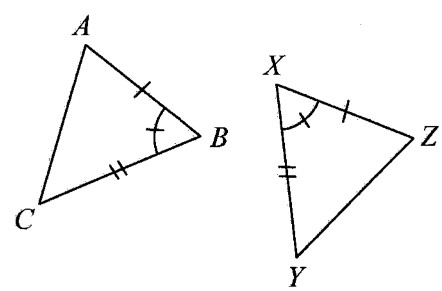


a. If $m\measuredangle x = 28°$, what is $m\overset{\frown}{AD}$?

b. If $AD = 5$ and $BD = 5\sqrt{3}$, what is the area of $\odot C$?

c. If the radius of $\odot C$ is 8 and if $m\overset{\frown}{BD} = 100°$, what is BD?

12-30. **Multiple Choice:** Based on the markings in the diagrams at right, which statement is true?



a. $\Delta ABC \cong \Delta XYZ$

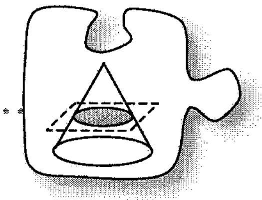
b. $\Delta ABC \cong \Delta YXZ$

c. $\Delta ABC \cong \Delta ZXY$

d. $\Delta ABC \cong \Delta ZYX$

e. None of these

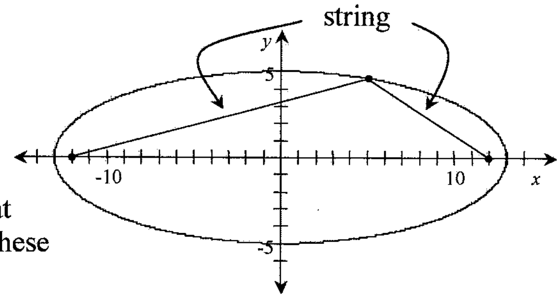
12.1.4 How can I construct it?



The Hyperbola

12-31. Review what you have learned about ellipses as you answer the questions below. You may find the information in the Math Notes box useful.

a. Maya used string and thumbtacks to draw the ellipse at right. How wide is her ellipse? How long is the string she used? How can you tell?



b. Maya's thumbtacks are placed at (12, 0) and (–12, 0). What are these points called?

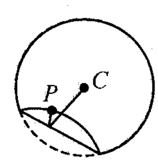
c. How could Maya change her thumbtacks so that the ellipse is 26 units tall and 10 units wide?

12-32. CONSTRUCTING AN ELLIPSE

Maya thinks that an ellipse is closely related to a circle. For one thing, they are both cross-sections of a cone. Also, when the two foci of an ellipse coincide (lie on top of each other), the ellipse becomes a circle. Therefore, Maya suspects that there must be a way to use a circle to construct an ellipse with tracing paper.

a. Use a compass to construct a circle with a radius of approximately 2 inches on a piece of tracing paper. Label its center C.

b. Find and label a point P inside the circle (other than the center). Then fold the tracing paper so that the circle passes through P. Unfold and then fold in a different location so that the circle still passes through P. Continue this process until an ellipse emerges.



c. Where are the foci of the ellipse?

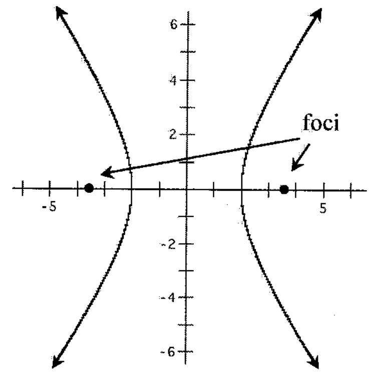
d. Why does this construction work? Pick a point on the circle and label it A. Draw the radius $\overline{AC}$. Study what happens as you fold the circle so that point A lies on point P. Explain why the sum of the distances between each point on the ellipse and the foci must be constant.

12-33. Maya asks, "What if the point is outside the circle?"

Your Task: Repeat the folding process that you used in problem 12-32 to explore Maya's question with your team. Each time you fold the tracing paper, be sure a different point on the circle passes through point P. Use fresh tracing paper for each construction. Write down what happens and be prepared to share your findings with the class.

12-34. THE HYPERBOLA

The shape you constructed in problem 12-33 is called a **hyperbola**. A hyperbola is sometimes described as having two curves facing away from each other.

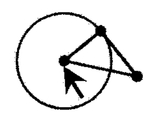


Example of a hyperbola

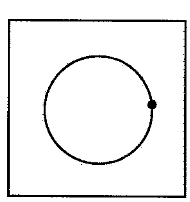
a. A hyperbola has two foci. **Examine** the hyperbola that you constructed in problem 12-33. Where do you think the foci might lie? Explain your thinking.

b. In an ellipse, the sum of the distances between each point on the ellipse and its foci must be constant. One way to remember this is to imagine constructing the ellipse with a string that is attached at both foci.

But what about a hyperbola? How are the distances between each point on the hyperbola and the foci related? Explore this idea with a dynamic geometry tool, if possible. (If a dynamic tool is not available, ask your teacher for the Lesson 12.1.4 Resource Page or download a copy from www.cpm.org.) Describe the relationship between the distances from each focus to each point on the hyperbola.



12-35. Maya is still wondering about her construction. "What if the point lies <u>on</u> the circle? What conic would be created then?" Use the dynamic geometry tool to **investigate** her question. Explain what happens.

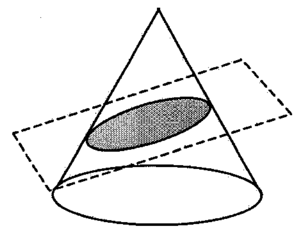


Methods and Meanings

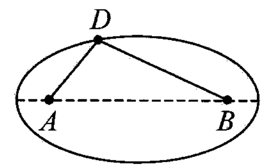
Math Notes

Ellipses

One of the cross-sections of a cone is an **ellipse**. An ellipse is often described as an oval or a circle that is "stretched."



An ellipse has two **foci** that lie on its longest line of symmetry. The sum of the distances between each point on the ellipse and the foci must be constant. For example, if A and B are the foci of the ellipse at right, then the sum of the lengths of $\overline{AD}$ and $\overline{BD}$ must be constant.

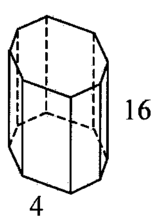


The sum of the distances between each point on the ellipse and the foci is also equal to the length of the widest distance across the ellipse.

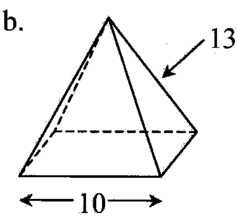
Review & Preview

12-36. Find the surface area of the solids below. Assume that the solid in part (a) is a prism with a regular octagonal base and the pyramid in part (b) is a square-based pyramid. Show all work.

a.



b.



12-37. Solve each equation below for the given variable. Check your solution by verifying that your solution makes the original equation true.

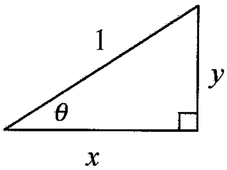
a. $\frac{2}{3}(15u-6)=14u$

b. $(5-x)(3x+8)=0$

c. $2(k-5)^2=32$

d. $2p^2+7p-9=0$

12-38. **Examine** the diagram at right.



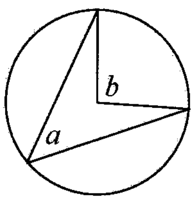
a. Explain why $y = \sin\theta$ and $x = \cos\theta$.

b. According to this diagram, what is $(\sin\theta)^2 + (\cos\theta)^2$? Explain how you know.

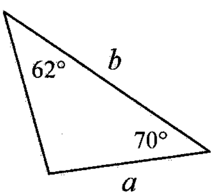
c. Is this relationship true for all angles? Use your calculator to find $(\sin 23°)^2 + (\cos 23°)^2$ and $(\sin 81°)^2 + (\cos 81°)^2$. Write down your findings.

12-39. For each geometric relationship below, determine whether a or b is larger, or if they are equal. Assume that the diagrams are not drawn to scale. If there is not enough information, explain what information is missing.

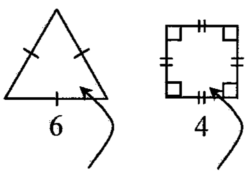
a.



b.



c.



area of the triangle is a. area of the square is b.

12-40. On graph paper, graph the function $y = x^2 - 3x - 4$.

a. What is y when $x = 3$? -2? $\frac{1}{2}$?

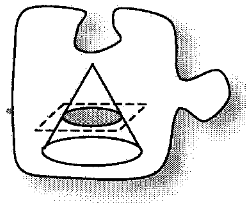
b. What is x when $y = -4$? 0?

12-41. **Multiple Choice:** What is the measure of each interior angle of a regular octagon?

a. 135° b. 120° c. 180° d. 1080°

12.1.5 What will it look like?

Conic Equations and Graphs



To complete this study of conics, today you will examine the types of equations that represent the different conic sections. You will review your understanding of equations for circles and parabolas and will extend your understanding to include equations for ellipses and hyperbolas. Then you will have an algebraic and geometric understanding of each of the conic sections.

12-42. How can you tell what an equation will look like when it is graphed? In this problem, you and your team will review what you already know about the equations of conic sections.

a. **Examine** the equations below. Which one is the equation of a line? Of a parabola? Of a circle? How can you tell?

(1) $x^2+y^2=25$ (2) $y=x^2-5$ (3) $y=x+4$

b. On a piece of graph paper, graph the equations at right. Then name all points of intersection in the form (x, y).

$x^2+y^2=25$

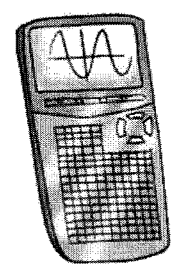
$y=x^2-5$

c. Is the circle you graphed in part (b) a function? What about the parabola? You may want to review the idea of "function" in the Math Notes box for this lesson.

12-43. GRAPH INVESTIGATION

Since the graph of the equation $x^2+y^2=4$ will be a circle with radius 2 and center at (0, 0), what happens when the equation is changed to become $5x^2+y^2=4$ or $x^2-2y^2=4$?

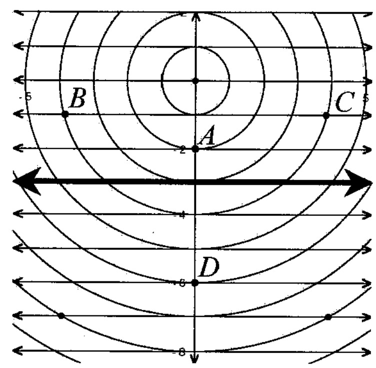
Your Task: Use a graphing tool such as a graphing calculator to investigate the graphs that can be found by altering the circle equation. Start with the equation $ax^2+by^2=1$ and find out what happens as you change the values of a and/or b. Write down any ideas or conjectures you find during this **investigation** and be ready to share them with the class.



12-44. GRAPHING HYPERBOLAS

In Lesson 12.1.2, you graphed a parabola by using the fact that each point on the graph was an equal distance to both the focus and directrix. Then, when you graphed a curve so that each point was closer to the focus than the directrix, you got an ellipse! What happens when each point is twice as far from the focus as it is from the directrix?

Obtain a sheet of focus-directrix paper from your teacher. Highlight the center of the circles to be the focus, and highlight a line that is three units away from the focus to be the directrix, as shown in the graph at right.



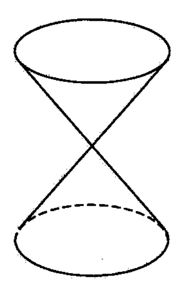
a. Points A, B, C, and D are all points that are twice as far from the focus than they are from the directrix. Plot these four points on your graph and confirm that the distance from each point to the focus is twice its distance to the directrix.

b. Continue plotting points that fit this pattern. Remember that each point you plot must be twice as far from the focus than the directrix. Also, notice in the case of point D above, the points can lie on both sides of the directrix. What curve appears?

12-45. RETURN TO THE CONE

The hyperbola seems linked to the conic sections you found in problem 12-1. For example, a hyperbola can be graphed using a focus and directrix, just like an ellipse and parabola. A hyperbola can also be constructed using a circle and a point outside a circle, much like the construction of an ellipse (which uses a circle and a point inside the circle). Even the equation of a hyperbola (such as $x^2 - y^2 = 1$) looks a lot like the equation of a circle or an ellipse.

The reason why a hyperbola did not appear when you found the cross-sections of a cone is because a hyperbola needs to have two branches curving away from each other. A way to get this cross-section is by using a **double-cone**, a shape created with two cones placed in opposite directions with vertices together, as shown at right. With your team, explain how to slice this double-cone to create a cross-section that is a hyperbola.



MATH NOTES

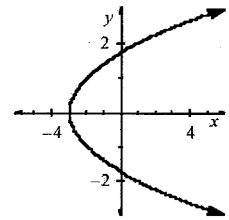
METHODS AND MEANINGS

Functions and Relations

A **relation** establishes a correspondence between its inputs and outputs (in math language called "sets"). For equations, it establishes the relationship between two variables and determines one variable when given the other. Some examples of relations are:

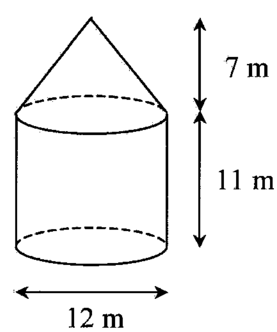
$$y = x^2,\ y = \frac{x}{x+3},\ y = -2x + 5$$

A **relation** is called a **function** if there exists no more than one output for each input. If a relation has two or more distinct outputs for a single input value, it is not a function. The relation graphed at right is not a function because, for example, there are two y-values for each x-value greater than –3.



Review & Preview

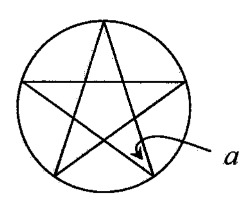
12-46. A silo (a structure designed to store grain) is designed as a cylinder with a cone on top, as shown in the diagram at right. Assume that the base of the cylinder is on the ground.



a. If a farmer wants to paint the silo, how much surface area must be painted?

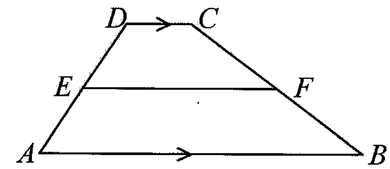
b. What is the volume of the silo? That is, how many cubic meters of grain can the silo hold?

12-47. A regular pentagram is a 5-pointed star that has congruent angles at each of its outer vertices. Use the fact that all regular pentagrams can be inscribed in a circle to find the measure of angle a at right.



12-48. In Chapter 7, you discovered that the midsegment of a triangle is not only parallel to the third side, but also half its length. But what about the midsegment of a trapezoid?

The diagram at right shows a midsegment of a trapezoid. That is, $\overline{EF}$ is a midsegment because points E and F are both midpoints of the non-base sides of trapezoid $ABCD$.



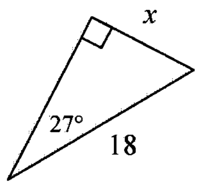
a. If $A(0,0)$, $B(9,0)$, $C(5,6)$, and $D(2,6)$, find the coordinates of E and F. Then compare the lengths of the bases ($\overline{AB}$ and $\overline{CD}$) with the length of the midsegment $\overline{EF}$. What seems to be the relationship?

b. See if the relationship you observed in part (a) holds if $A(-4,0)$, $B(2,0)$, $C(0,2)$, and $D(-2,2)$.

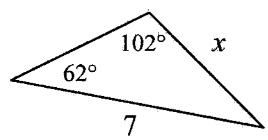
c. Write a conjecture about the midsegment of a trapezoid.

12-49. For each diagram below, solve for x. Show all work.

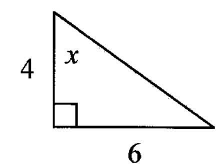
a.



b.

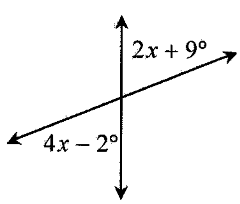


c.

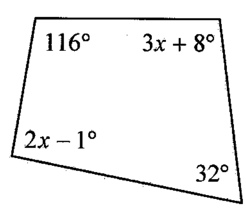


12-50. For each relationship below, write and solve an equation for x. **Justify** your method.

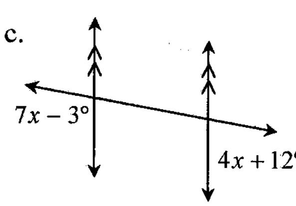
a.



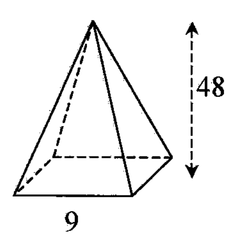
b.



c.



12-51. **Multiple Choice:** The volume of the square-based pyramid with base edge 9 units and height 48 units is:



a. 324 un^3 b. 1296 un^3

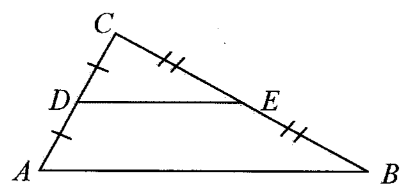
c. 3888 un^3 d. not enough information

12.2.1 What's the shape?

Using Coordinate Geometry and Construction to Explore Shapes

In today's activity, you will learn more about quadrilaterals as you review what you know about coordinate geometry, construction, and proof.

12-52. Review what you have learned about the midsegment of a triangle as you answer the questions below. Assume that $\overline{DE}$ is a midsegment of $\triangle ABC$.



a. What is the relationship between $\overline{DE}$ and $\overline{AB}$?

b. What is the relationship between $\measuredangle CDE$ and $\measuredangle DAB$? How do you know?

c. What is the relationship between $\triangle ABC$ and $\triangle DEC$? **Justify** your conclusion.

d. If $DE = 4x + 7$ units and $AB = 34$ units, what is x?

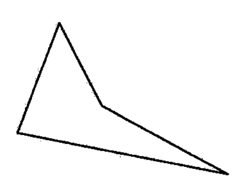
12-53. QUIRKY QUADRILATERALS

Quinn decided to experiment with the midpoints of the sides of a quadrilateral one afternoon. With a compass, he located the midpoint of each side of a quadrilateral. He then connected the four midpoints together to create a new quadrilateral inside his original quadrilateral.

a. Without knowing anything about Quinn's original quadrilateral and without trying the construction yourself, **visualize** the result. What can you predict about Quinn's resulting quadrilateral? Share your ideas with your team.

b. Use a compass and straightedge to repeat Quinn's experiment on an unlined piece of paper. Make sure each member of your team starts with a differently-shaped quadrilateral. Describe your results. Did the results of you and your teammates match your prediction from part (a)?

c. Does it matter if your starting quadrilateral is convex or not? Start with a non-convex quadrilateral, like the one shown at right, and repeat Quinn's experiment. On your paper, describe your results.



12-54. Quinn decided to graph his quadrilateral on a set of coordinate axes and prove that his inner quadrilateral is, in fact, a parallelogram. His quadrilateral *ABCD* uses the points $A(-3, -2)$, $B(-5, 4)$, $C(5, 6)$, and $D(1, -4)$.

a. On graph paper, graph the quadrilateral *ABCD*.

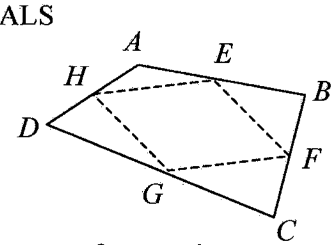
b. If the midpoint of $\overline{AB}$ is *E*, the midpoint of $\overline{BC}$ is *F*, the midpoint of $\overline{CD}$ is *G*, and the midpoint of $\overline{DA}$ is *H*, find and label points *E, F, G,* and *H* on *ABCD*.

c. Connect the midpoints of the sides you found in part (b). Then find the slope of each side of quadrilateral *EFGH* and use these slopes to prove that Quinn's inner quadrilateral is a parallelogram.

d. Quinn wondered if his parallelogram is also a rhombus. Find *EF* and *FG*, and then decide if *EFGH* is a rhombus. Show all work.

12-55. PROVING THE RESULT FOR ALL QUADRILATERALS

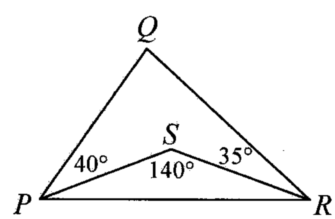
In problem 12-54, you proved that Quinn's inner quadrilateral was a parallelogram when $A(-3, -2)$, $B(-5, 4)$, $C(5, 6)$, and $D(1, -4)$. However, what about other, random quadrilaterals?



To prove this works for all quadrilaterals, start with a diagram of a generic quadrilateral, like the one above. Assume that the midpoint of $\overline{AB}$ is *E*, the midpoint of $\overline{BC}$ is *F*, the midpoint of $\overline{CD}$ is *G*, and the midpoint of $\overline{DA}$ is *H*. Prove that *EFGH* is a parallelogram by proving that its opposite sides are parallel. It may help you to draw diagonal $\overline{AC}$ and consider what you know about ΔABC and ΔACD. Use any format of proof.

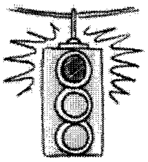
Review & Preview

12-56. In ΔPQR at right, what is $m\measuredangle Q$? Explain how you found your answer.

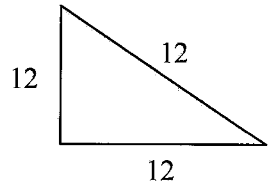


12-57. The United State Department of Defense is located in a building called the Pentagon because it is in the shape of a regular pentagon. Known as "the largest office building in the world," it's exterior edges measure 921 feet. Find the area of land enclosed by the outer walls of the Pentagon building.

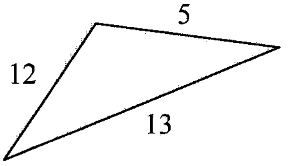
12-58. **Examine** the triangles below. Decide if each one is a right triangle. If the triangle is a right triangle, **justify** your conclusion. Assume that the diagrams are not drawn to scale.



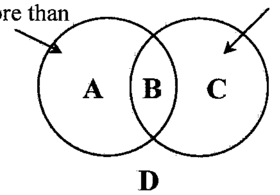
a.

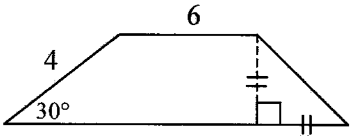


b.



12-59. **Examine** the Venn diagram at right. In which region should the figure below be placed? Show all work to **justify** your conclusion.





12-60. For each equation below, decide if the equation has any real number for the given variable. For each problem, explain how you know.

a. $4(x-3)=11$

b. $x^2=-10$

c. $3x^2-18=0$

d. $-7=|x-6|$

12-61. **Multiple Choice:** Which number below could be the length of the third side of a triangle with sides of length 29 and 51?

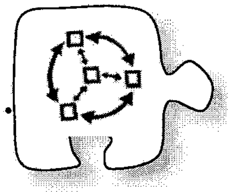
a. 10

b. 18

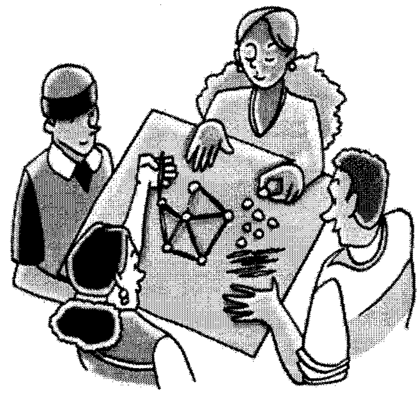
c. 23

d. 81

12.2.2 What's the pattern?



Euler's Formula for Polyhedra



Throughout this course, you have developed your skills of exploring a pattern, forming a conjecture, and then proving your conjecture. Today, with the assistance of materials such as toothpicks and gumdrops, you will review what you know about basic polyhedra (with no holes) as you look for a relationship between the number faces, edges, and vertices each basic polyhedron has. Once you have written a conjecture, your class will discuss how to prove that it must be true.

As you work today, consider the following questions:

Which types of basic polyhedra have we not tested yet?

What relationship can I find between the number of edges, faces, and vertices of a basic polyhedron?

Does this relationship always hold true?

12-62. POLYHEDRA PATTERNS

Does a basic polyhedron (with no holes) usually have more faces, edges, or vertices? And if you know the number of faces and vertices of a basic polyhedron, how can you predict the number of edges? Today you will answer these questions and more as you **investigate** polyhedra.

Your Task: Obtain the necessary building materials from your teacher, such as toothpicks (for edges) and gumdrops (for vertices). Your team should build *at least* six distinctly different polyhedra and each person in your team is responsible for building *at least* one polyhedron. Be sure to build some regular polyhedra (such as a tetrahedron and an octahedron), basic prisms, pyramids, and unnamed polyhedra.

Create a table like the one at right to hold your data. Once you have recorded the number of vertices, edges, and faces for your team's polyhedra, look for a relationship between the numbers in each row of the table. Try adding, subtracting (or both) the numbers to find a pattern. Write a conjecture (equation) using the variables F, V, and E.

Polyhedron	Faces (F)	Vertices (V)	Edges (E)

12-63. EULER'S FORMULA FOR POLYHEDRA

The relationship you discovered in problem 12-62 between the number of faces, vertices, and edges of a basic polyhedron is referred to as **Euler's Formula for Polyhedra**, after Leonhard Euler (pronounced "oiler"), a mathematician from Switzerland.[1] It states that if V is the number of vertices, F is the number of faces, and E is the number of edges of a basic polyhedron, then $V+F-E=2$.

Use Euler's Formula to answer the following questions about basic polyhedra.

a. If a polyhedron has 5 faces and 6 vertices, how many edges must it have?

b. What if a polyhedron has 36 edges and 14 faces? How many vertices must it have?

c. Could a polyhedron have 10 faces, 3 vertices, and 11 edges? Explain why or why not.

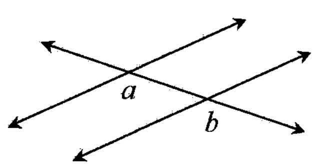
12-64. If V represents the number of vertices, F represents the number of faces, and E represents the number of edges of a basic polyhedron, how can you prove that $V+F-E=2$? First think about this independently. Then, as a class, prove Euler's Formula.

Review & Preview

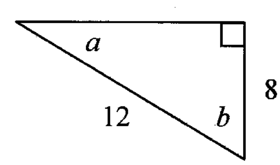
12-65. For each situation below, decide if a is greater, b is greater, if they are the same value, or if not enough information is given.

a. a is the measure of a central angle of an equilateral triangle; b is the measure of an interior angle of a regular pentagon.

b.

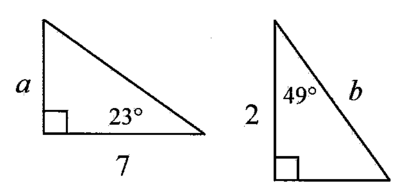


c.



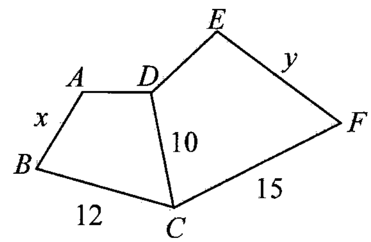
d. $a=b+3$

e.



[1] While it is widely believed that Euler independently discovered this relationship, it has been recorded that René Descartes (pronounced "Day-cart") found the relationship over 100 years earlier.

12-66. In the diagram at right, $ABCD \sim DCFE$. Solve for x and y. Show all work.



12-67. On graph paper, make a table and graph the function $f(x) = -2(x-1)^2 + 8$.

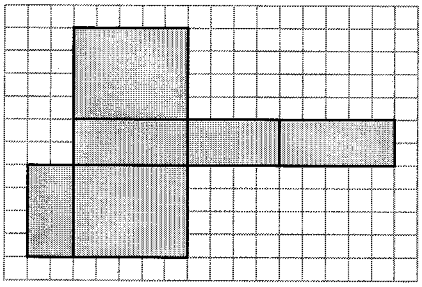
a. Label the x- and y-intercepts and state their coordinates.

b. Name the vertex.

c. Find $f(100)$ and $f(-15)$.

12-68. Find the area of the graph of the solution region of $x^2 + y^2 \leq 49$.

12-69. Find the volume and surface area of the box formed by the shaded net at right.



12-70. **Multiple Choice:** The radius of the front wheel of Gavin's tricycle is 8 inches. If Gavin rode his tricycle for 1 mile in a parade, approximately how many rotations did his front wheel make? (Note: 1 mile = 5280 feet).

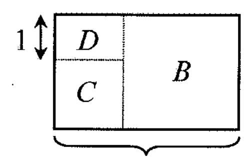
a. 50 b. 1260 c. 660 d. 42,240

12.2.3 What's special about this ratio?

The Golden Ratio

In Chapter 8, you discovered an important irrational number: π. Pi (π) is the ratio of any circle's circumference to its diameter. However, there is another special ratio that appears not only in geometry, but also in nature. Today, you will discover this number and will examine several different contexts in which this number appears.

12-71. While doodling one day, Alex drew the diagram at right. He started with a large rectangle (A). He divided this rectangle into a square (B) and a smaller rectangle. Then he divided the smaller rectangle into a square (C) and a rectangle (D). He noticed that D had a height of 1 unit (as shown in the diagram).



A is the entire rectangle.

a. Draw Alex's diagram on a piece of graph paper.

b. If the length of each side of square C is 3 units, what is the area of square B? What are the dimensions of rectangle A? **Justify** your answers.

c. If the longest dimension of rectangle A is 9 units, what are the dimensions of squares B and C? Show how you know.

12-72. THE GOLDEN RATIO

As Alex looked at his diagram from problem 12-71, he noticed that the dimensions of the large composite rectangle seemed to be subdivided proportionally. In other words, Figures E and F, shown at right, appeared to be similar.

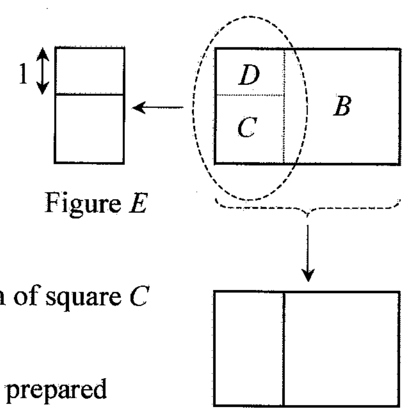


Figure *E*

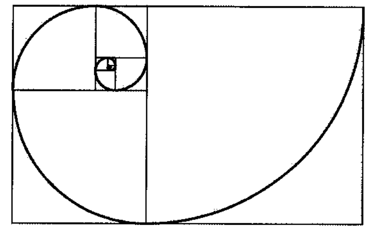
Figure *F*

a. If the height of D is 1 unit and the side length of square C is x, what is the side length of square B?

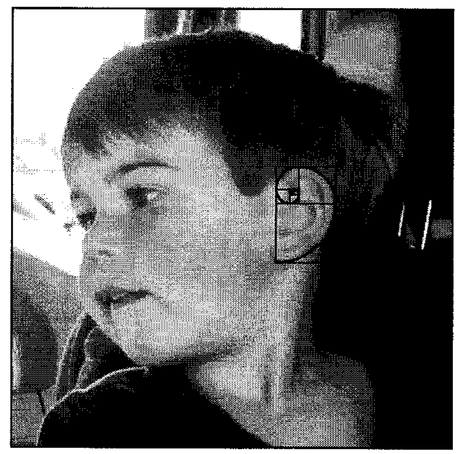
b. If Figures E and F are similar, what is x? Be prepared to share your solution with the class.

c. The value for x that you found in part (b) has a special name: the **Golden Ratio**. It is often represented by the greek letter phi (ϕ), pronounced "fee." Read the Math Notes box about the golden ratio before moving on to problem 12-73.

12-73. GOLDEN SPIRALS


Golden spiral

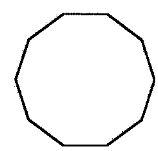
Each non-square rectangle in Alex's diagram from problem 12-71 is an example of a **golden rectangle** because the ratio of the longer length to the shorter length is ϕ, the golden ratio. In addition, Alex's process of subdividing each golden rectangle into a square and smaller golden rectangle can be **iterated** (repeated over and over) creating an infinite series of nested squares and rectangles.



When connected arcs are placed in each of the squares, a spiral forms, like the one shown above. One place you can find this spiral is the human ear, as shown at right.

Use a compass to draw a golden spiral on the Activity 12.2.3 Resource Page provided by your teacher. Where else (outside of class) have you seen a spiral like this?

12-74. Alex wonders where else the number phi (ϕ) shows up. Look for phi (ϕ) as you analyze the following situations.



a. **Examine** the regular decagon at right. If the side length is 1 unit, find the radius of the decagon. What do you notice?

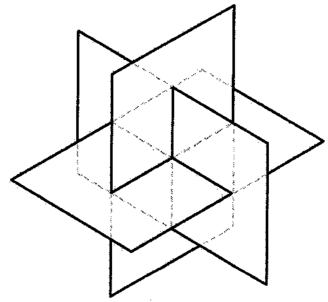
b. Each central triangle in the regular decagon from part (a) is a **golden triangle** because the ratio of the congruent sides to the base of each triangle is phi (ϕ). What are the angles of a golden triangle?

c. In problem 12-73, you learned about nested golden rectangles (where each golden rectangle is subdivided into a square and a smaller golden rectangle). But what about nested expressions?

Consider the expression at right. The "..." signifies that the pattern within the expression continues infinitely. With your team, find a way to approximate the value of this expression. Try to find the most accurate approximation you can. What do you notice?

$$1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

12-75. What if three golden rectangles intersect perpendicularly so that their centers coincide, as shown at right? If each vertex of the golden rectangles is connected with the five closest vertices, what three-dimensional shape appears? First **visualize** the result. Then, if you have a model available, test your idea with string.



MATH NOTES

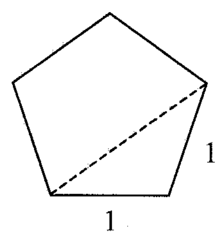
LOOKING DEEPER

The Golden Ratio

The number phi (ϕ), pronounced "fee," has a value of $\frac{1+\sqrt{5}}{2} \approx 1.618$ and is often referred to as the **golden ratio**. This special number is often found when comparing dimensions of geometric shapes and by comparing measurements of objects in nature.

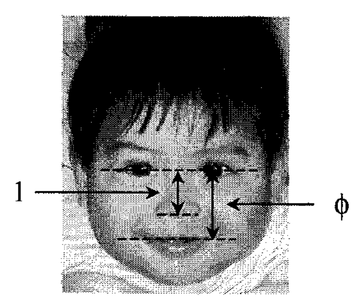
For example, phi can be found multiple ways in a regular pentagon. The ratio of the length of any diagonal of a regular pentagon to the length of a side is phi (ϕ). This can be shown by assuming the side length is 1 unit and finding the length of the diagonal. Since each interior angle of the pentagon must be 108°, then the length of the diagonal must be:

$$d = \sqrt{1^2 + 1^2 - 2(1)(1)\cos 108°} \approx 1.618$$



Regular pentagon

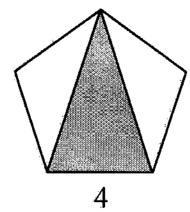
Phi also appears in nature. For example, the human body has many ratios that seem to relate to phi (ϕ). As shown in the picture at right, the ratio of the distance between the eyes and the corners of the mouth to the length of the nose is often phi (ϕ). That is, if the nose is 1 unit long, the vertical distance between the eyes and the corners of the mouth is often $\frac{1+\sqrt{5}}{2} \approx 1.618$ units.



Review & Preview

12-76. If ΔABC is equilateral, and if $A(3,2)$ and $B(7,2)$, find all possible coordinates of vertex C. **Justify** your answer.

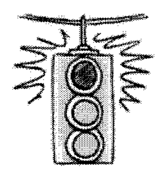
12-77. Find the area of the shaded region of the regular pentagon at right. Show all work.



12-78. On graph paper, graph the following system of inequalities. Be sure that your shaded region represents all of the points that make both inequalities true.

$$y < \tfrac{2}{3}x - 2$$
$$y \geq -5x - 2$$

12-79. Jamila solved the quadratic $x^2 + 3x - 10 = 8$ (see her work below). When she checked her solutions, they did not make the equation true. However, Jamila cannot find her mistake. Explain her error and then solve the quadratic correctly.



$$x^2 + 3x - 10 = 8$$
$$(x+5)(x-2) = 8$$
$$x + 5 = 8 \quad \text{or} \quad x - 2 = 8$$
$$x = 3 \quad \text{or} \quad x = 10$$

12-80. If the sum of the interior angles of a regular polygon is 2160°, how many sides must it have?

12-81. **Multiple Choice:** Assume that $A(6,2)$, $B(3,4)$, and $C(4,-1)$. If ΔABC is rotated 90° counterclockwise (↺) to form $\Delta A'B'C'$, and if $\Delta A'B'C'$ is reflected across the x-axis to form $\Delta A''B''C''$, then this is the coordinates of C''.

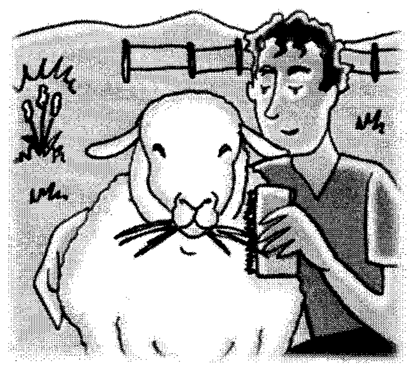
a. (1, 4) b. (–4, 1) c. (1, –4) d. (4, 1)

12.2.4 What's the probability?

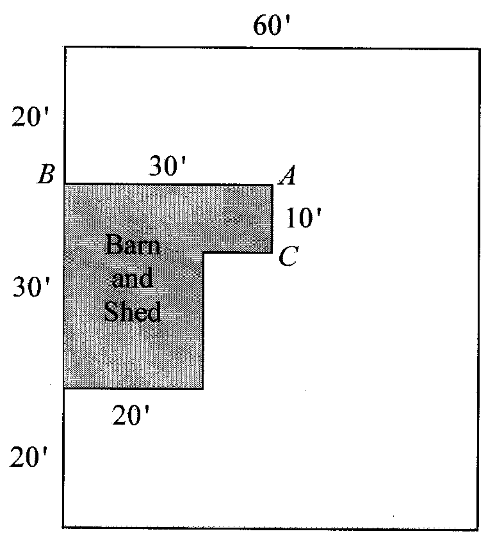
Using Geometry to Find Probability

In this final activity, you will **connect** and **apply** much of your knowledge from throughout the course to solve a challenging problem.

12-82. ZOE AND THE POISON WEED



Dimitri is getting his prize sheep, Zoe, ready for the county fair. He keeps Zoe in the pasture beside the barn and shed. What he does not know is that there is a single locoweed in this pasture, which will make Zoe too sick to go to the fair if she eats it, and she can eat it in one bite. Zoe takes about one bite of grass or plant every three minutes for six hours a day.



The layout of the field and building is provided at left and on the Activity 12.2.4 Resource Page provided by your teacher. Assume that the entire field (the unshaded region) has plants growing on it and that each square foot of field provides enough food for 40 bites. Also assume that each corner of the barn and field is a right angle.

Dimitri is worried that Zoe will get into trouble unless she is tethered with a rope to the building. He has decided to tether Zoe at point *A* with a 20-foot rope. Zoe is unable to enter the barn or shed while on her tether.

Your Task: If the locoweed lies in Zoe's grazing area, what is the probability that Zoe will get sick in one day?

Discussion Points

What is the problem asking you to find?

What does Zoe's grazing region look like?

What do you need to figure out in order to find the probability?

12-83. To help find the probability that Zoe will eat the single locoweed, first consider the grazing region if she is tethered to point A with a 20-foot rope.

a. On your Activity 12.2.4 Resource Page, draw and label the region that Zoe can roam. Then find the area of that region.

b. Since each square foot of the field contains 40 bites of food, how many bites of food lie within Zoe's reach?

c. How many bites of food does Zoe eat each day? Show your calculations.

d. If the single locoweed is within this area, what is the probability that she eats the weed in one day? Be prepared to explain your answer to the class.

Further Guidance section ends here.

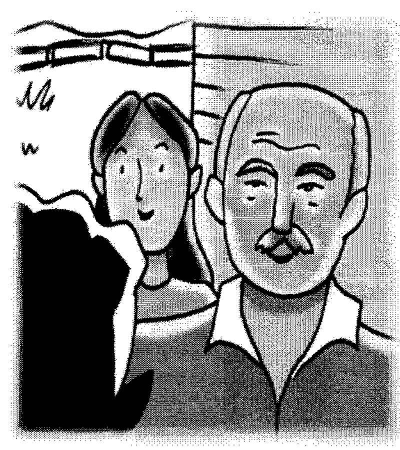
12-84. FAMILY DISCUSSION

When Dimitri discussed his idea with his family, he received other ideas! Analyze each of the ideas given below and then report back to Dimitri about which of them has the least probability that Zoe will eat the locoweed. Your analysis should include:

- A diagram of each proposed region on the Activity 12.2.4 Resource Page (or use the figure in problem 12-82).
- All calculations that help you determine the probability that Zoe will eat the poisoned weed for each proposed region.

Assume that a single locoweed lies somewhere in each region that is proposed.

a. **Dimitri's Father:** "Dimitri! Why do you need to waste rope? All you need is to tether your sheep with a 10-foot rope attached at point A. Take it from me: Less area to roam means there is less chance that the sheep will eat the terrible locoweed!"



b. **Dimitri's Sister:** "I don't agree. I think you should consider using a 30-foot rope attached to point B. The longer rope will give Zoe more freedom, but the building and fences will still limit her region. This is the best way to reduce the chance that Zoe gets sick before the fair."

Problem continues on next page →

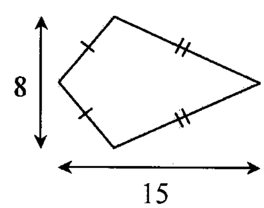
12-84. *Problem continued from previous page.*

c. **Dimitri's Mother:** "Both of those regions really restrict Zoe to the north-eastern part of the field. That means she won't be able to take advantage of the grass grown in the southern section of the field that is rich in nutrients because of better sunlight. I recommend that you use a 30-foot rope attached to point *C*. You won't be disappointed!"

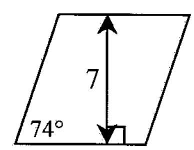
Review & Preview

12-85. Find the area of each quadrilateral below. Show all work.

a. Kite



b. Rhombus



12-86. A spinner is divided into two regions. One region, red, has a central angle of 60°. The other region is blue.

a. On your paper, sketch a picture of this spinner.

b. If the spinner is spun twice, what is the probability that both spins land on blue?

c. If the radius of the spinner is 7 cm, what is the area of the blue region?

d. A different spinner has three regions: purple, mauve, and green. If the probability of landing on purple is $\frac{1}{4}$ and the probability of landing on mauve is $\frac{2}{3}$, what is the central angle of the green region?

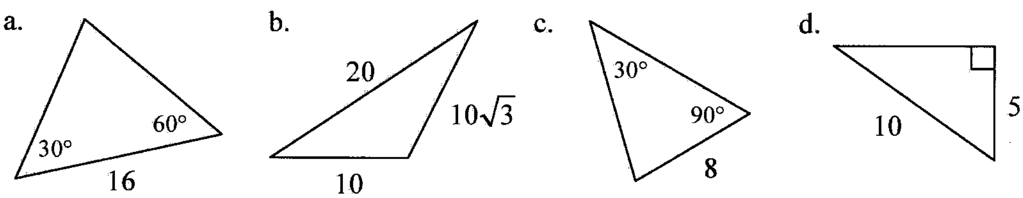
12-87. Perry threw a tennis ball up into the air from the edge of a cliff. The height of the ball was $y = -16x^2 + 64x + 80$, where y represents the height in feet of the ball above ground at the bottom of the cliff, and x represents the time in seconds after the ball is thrown.

a. How high was the ball when it was thrown? How do you know?

b. What was the height of the ball 3 seconds after it was thrown? What was its height $\frac{1}{2}$ a second after it was thrown? Show all work.

c. When did the ball hit the ground? Write and solve an equation that represents this situation.

12-88. **Examine** the triangles below. Which, if any, are similar? Which are congruent? For each pair that must be similar, state how you know. Remember that the diagrams are not drawn to scale.



12-89. **Examine** the mat plan of a three-dimensional solid at right.

3	1	0
1	2	1
0	2	1

RIGHT

FRONT

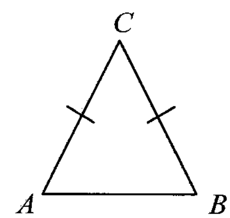
Mat Plan

a. On your paper, draw the front, right, and top views of this solid.

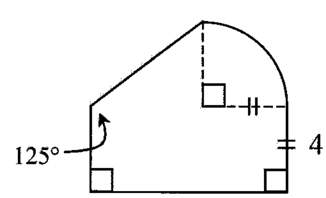
b. Find the volume of the solid.

c. If each edge of the solid is multiplied by 5, what will the new volume be? Show how you got your answer.

12-90. Prove that the base angles of an isosceles triangle must be congruent. That is, prove that if $\overline{BC} \cong \overline{AC}$ in the triangle at right, then $\measuredangle A \cong \measuredangle B$. (Hint: Is there a convenient auxiliary line that can be added to the diagram that divides the triangle into two congruent triangles?)

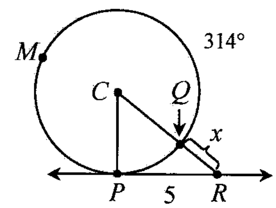


12-91. Find the area and perimeter of the shape at right. Assume that any non-straight portions of the shape are part of a circle. Show all work.

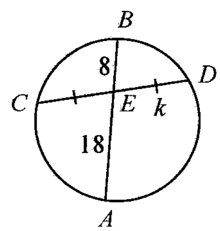


12-92. **Examine** the diagrams below. For each one, use the geometric relationships to solve for the given variable.

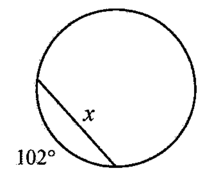
a. $\overleftrightarrow{PR}$ is tangent to $\odot C$ at P and $m\overparen{PMQ} = 314°$. Find QR.



b. $\overline{AB}$ and $\overline{CD}$ intersect at E.



c. Radius = 7 cm

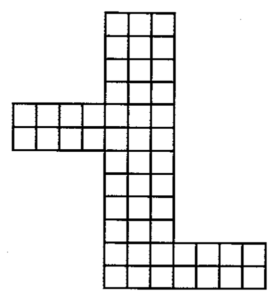


12-93. **Examine** the net at right.

a. Describe the solid that is formed by this net. What are its dimensions?

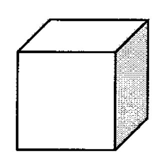
b. Find the surface area and volume of the solid formed by this net.

c. If all the dimensions of this solid are multiplied by 3, what is the SA of the resulting solid? What is the volume?



12-94. Polly has a pentagon with measures $3x-26°$, $2x+70°$, $5x-10°$, $3x$, and $2x+56°$. Find the probability that if one vertex is selected at random, then the measure of its angle is more than or equal to 90°.

12-95. Find at least three different shapes that can be cross-sections of a cube, like the one at right. For each one, draw the resulting cross-section and explain how you sliced the cube.



12-96. **Multiple Choice:** A square based pyramid has a slant height of 10 units and a base edge of 10 units. What is the height of the pyramid?

a. 5 b. $5\sqrt{3}$ c. 6 d. 8