14.6 Notes Day 1: Transformation Matrices

Transformation matrices are used to apply linear transformations (i.e., reflections, translations, rotations, & dilations) to matrices that represent linear systems.

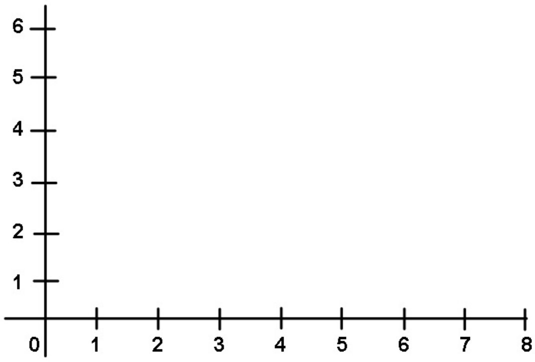
Recall: Given linear transformation T:(x, y) → (3x-2y, x+y), to find T:(7, 4) we plug in 7 for x and 4 for y to get T:(7, 4) → (3(7)-2(4), 7+4) = (13, 11)

ex 1: Given ΔABC with vertices A(1,1), B(2,3), & C(4,2)

a) Graph ΔABC

b) Find the image under T:(x, y) → (3x-2y, x+y)

A' = T:(1,1) → ( )
B' = T:(2,3) → ( )
C' = T:(4,2) → ( )



c) Graph ΔA'B'C'

How do the sizes of the triangles compare? Δ ______ is bigger.

Do the Δs have the same orientation (both names clockwise)? Y / N

d) Repeat part (b) using a transformation matrix for 3x-2y and x+y

T:(x,y) = [[ ] [[x]
[ ]] [y]]
thus transformation matrix T is... T = [[ ]
[ ]]

T:(1,1) = [[ ] [[ ] = [[ ] = [[ ]
[ ]] [ ]] [ ]] [ ]]

T:(2,3) = [[ ] [[ ] = [[ ] = [[ ]
[ ]] [ ]] [ ]] [ ]]

T:(4,2) = [[ ] [[ ] = [[ ] = [[ ]
[ ]] [ ]] [ ]] [ ]]

e) Find the ratio of the area of the new triangle to the original triangle.

<u>area of ΔA'B'C'</u> = absolute value of the determinant of T
area of ΔABC

= | | = | | = | | =

∴ The new triangle is the area of the original triangle.

f) Is the new triangle's orientation the same as the original's or different?

ex 2: Given T:(x, y) → (x-2y, 2y)

a) Find transformation matrix T: T = [[ ]
[ ]]

b) Find the determinant of T: | T | = =

c) Find the image of A(1,-1), B(2,0), & C(-3,1) under transformation T:

T		A	B	C		A'	B'	C'
[[ ]	[[			]	=	[[		]
[ ]]	[			]]		[		]]

↘ ↘ ↘

A'( ) B'( ) C'( )

d) How do the areas of triangles ABC & A'B'C' compare?

Image ΔA'B'C' is __________ the area of ΔABC

Preimage (original) ΔABC is __________ the area of ΔA'B'C'

e) Is the orientation of the image the same as or the opposite of the original? Why?

The orientation is __________ b/c ____________________

ex 3: Composition of transformations: Given transformation matrices

S = [[2 3] [-1 5]] and T = [[0 1] [2 -1]]

and **point P(3,4)**, find each of the following:

a) (T)(P) = [[0 1] [2 -1]] [[ ] [ ]] = [[ ] [ ]]

b) (S)(TP) = [[ ] [ ]] [[ ] [ ]] = [[ ] [ ]]

c) Use the associativity of matrix multiplication to find ST(P) via ONE transformation instead of two as we did in parts (a) and (b).

ST = [[2 3] [-1 5]] [[0 1] [2 -1]] = [[ ] [ ]]

(ST)(P) = [[ ] [ ]] [[ ] [ ]] = [[ ] [ ]]

Recall: ST is called a "composite" transformation S ∘ T

Note: On your HW, to "describe the motion" of a transformation, give the motion left ___ or right ___ (i.e., delta x) & the motion down ___ or up ___ (i.e., delta y). Δx = change in the x direction ↗

↖ Δy = change in the y direction

14.5 HW **page 535 # 26b** Manufacturing

	tables	chairs	desks
Carpentry	[[120	105	125]
Assembly	[40	65	110]
Finishing	[80	90	125]]

amnt. of time available	
Carpentry	[[14,960]
Assembly	[8,970]
Finishing	[12,590]]

Want to use all available labor. How many t, c, d, should the manager schedule for production each week? I.e., want to find t, c, d.

A **X** **=** **B**

[[120 105 125] [[t] [[14,960]
[40 65 110] [c] = [8,970]
[80 90 125]] [d]] [12,590]]

$\mathbf{X} = \mathbf{A^{-1} B}$ ← Do on T.I.

X = [[t] [c] [d]] = [[18] [110] [10]]

18 tables, 110 chairs, 10 desks

14.5 HW **page 546 # 5** Weather Forecasting

	sunny	cloudy	rainy
sunny	[[.80	.10	.10]
cloudy	[.00	.40	.60]
rainy	[1.00	.00	.00]]

= T ← (a) Day-to-day weather transition matrix

(b) Calculate T^2

On T.I., enter 3x3 matrix using Matrix >> Edit
Then calculate T^2 by using Matrix >> Names
2nd >> Quit >> Use x^2 key

T^2 =
[[.74 .12 .14]
[.60 .16 .24]
[.80 .10 .10]]

← Finding weather for two days from current day

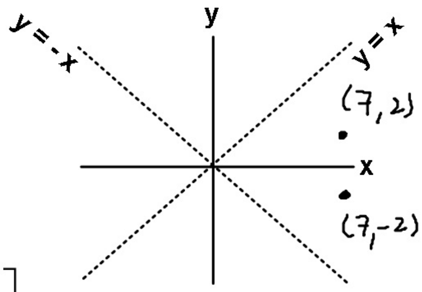
M_0 = [1.00 .00 .00] ← Initial weather values on Monday

To find weather on Wednesday given Monday, find: $M_0 T^2$

= [.74 .12 .14]

Thus we have a 14% chance of it being rainy.

14.6 Notes Day 2: Specific Transformations

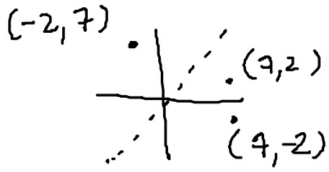


ex 1: Graph the point (7, 2). Then...

Reflect Over...	New Point	Transformation Matrix ?
a) the x-axis	(7, -2)	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ $R_{x\text{-axis}}$ ↗
b) the y-axis	(-7, 2)	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$ $R_{y\text{-axis}}$ ↗
c) line y = x	(2, 7)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ $R_{y=x}$ ↗
d) line y = -x	(-2, -7)	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$ $R_{y=-x}$ ↗

e) Find $(R_{y=x})(R_{x\text{-axis}})$ using the transformation matrices we found:

$$(R_{y=x})(R_{x\text{-axis}}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



f) Verify on your graph that this matrix works using sample point (7, 2); show how point (7, 2) changes using...

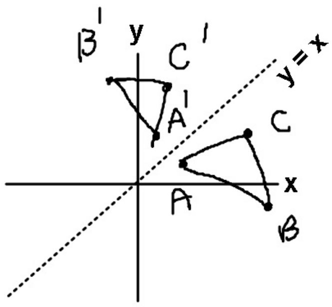
The graph: (7, 2) flipped over the x-axis becomes (7, -2);
(7, -2) flipped over line y = x becomes (-2, 7)

* Note: we apply the transformation for the matrix closest to the input first; x-axis and THEN y = x

The matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

Matrix from part (e) ↗

ex 2: Graph preimage ΔABC with vertices A(2,1), B(7,-1), & C(6,2).
Then apply the transformation matrix $R_{y=x}$ & graph the image.

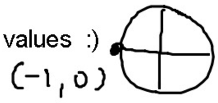


$$\underset{R_{y=x}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{A\quad B\quad C}{\begin{bmatrix} 2 & 7 & 6 \\ 1 & -1 & 2 \end{bmatrix}} = \underset{A'\quad B'\quad C'}{\begin{bmatrix} 1 & -1 & 2 \\ 2 & 7 & 6 \end{bmatrix}}$$

Thm: $R_{\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ will apply a θ-degree rotation about the origin.

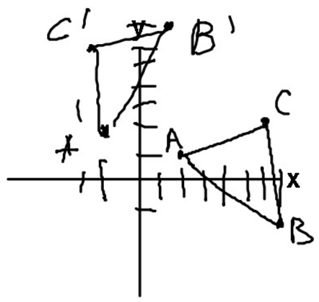
ex 3: Find:

a) $R_{180°} = \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ← Unit circle values :)



b) $(R_{180°})(R_{x\text{-axis}}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ = R <u>y-axis</u>

ex 4: Graph preimage ΔABC with vertices A(2,1), B(7,-1), & C(6,2).
Then find and graph its image under transformation $R_{90°}$.



Step 1: $R_{90°} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Step 2: $(R_{90°})(\Delta ABC) = \Delta A'B'C'$

$$\underset{R_{90°}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \underset{A\quad B\quad C}{\begin{bmatrix} 2 & 7 & 6 \\ 1 & -1 & 2 \end{bmatrix}} = \underset{A'\quad B'\quad C'}{\begin{bmatrix} -1 & 1 & -2 \\ 2 & 7 & 6 \end{bmatrix}}$$

↘ A'(-1, 2)
B'(1, 7)
C'(-2, 6)

Chapter 14 Study Guide

- You may use a 3 x 5 notecard of your own hand-written formulae on the Ch. 14 Test.
- Recommended: Review WS (make sure you understand the linguistics)
- Need to know:
  - Properties of matrices (the chart in your notes)
  - Matrix addition, subtraction, & multiplication (by hand)
  - Scalar multiplication (by hand)
  - Determinant
  - Dimensions
  - Notes & HW applications, e.g. communication matrices (given direct communication, i.e. no relay, with matrix M, M^2 means you use exactly 1 relay, $M + M^2$ means at most 1 relay, M^3 means you use exactly 2 relays, $M + M^2 + M^3$ means at most 2 relays, etc.)
  - Transition matrices (given initial state M_0, find next state $M_1 = M_0T$, $M_2 = M_0T^2$, $M_3 = M_0T^3$, et cetera; find steady state S s.t. $S = S\,T$)
  - Transformation matrices (given transformation T and coordinate points, build a matrix eq'n to find the image of the points -- i.e., [T] [matrix of preimage points] = [matrix of image points])
  - Et cetera

Extra review examples:

1) Write a matrix equation to find the intersection of the three planes (do not solve unless you have a T.I.):

$$\begin{aligned} 3x - 4y + 7z &= -3 \\ 6x + 3y + 5z &= 23 \\ -4x + 9y + 37z &= -17 \end{aligned}$$

Step 1: Write matrix eq'n using coefficients of linear system

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$$

If you were asked to solve on the test, you would be given a 2x2 so that you could compute the determinant & the inverse by hand.

Step 2: IF you have a T.I., then find the inverse of the matrix on the far LHS; multiply this inverse on the LEFT on both sides of the eq'n to get **answer** written as a coordinate point: (x, y, z) = **(3.29, 2.13, -.62)**

2) Given $A = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$, find the following (by hand):

a) A^t b) A^{-1} c) A^2

Pop Quiz Ch. 14 /17 NO calculators!

① $\begin{bmatrix} 3x & 4y \\ 8z & 2w \end{bmatrix} = \begin{bmatrix} 15 & 20 \\ 80 & 142 \end{bmatrix}$ Solve

② $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 4 \\ 5 & 9 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 4 & 0 & -1 \\ 10 & 5 & 0 \end{bmatrix} =$

③ Solve $\begin{bmatrix} 3 & 5 \\ 8 & 4 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Pop Quiz Ch. 14 score/17

① $\begin{bmatrix} 3x & 4y \\ 8z & 2w \end{bmatrix} = \begin{bmatrix} 15 & 20 \\ 80 & 142 \end{bmatrix}$ Solve $x=5$ $y=5$ $z=10$ $w=71$

/4

② $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 4 \\ 5 & 9 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 4 & 0 & -1 \\ 10 & 5 & 0 \end{bmatrix} =$ $\begin{bmatrix} 64 & 25 & -1 \\ 41 & 20 & 1 \\ 121 & 40 & -4 \end{bmatrix}$

/9

③ Solve $\begin{bmatrix} 3 & 5 \\ 8 & 4 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

$X = \begin{bmatrix} -3/28 & -1/7 \\ 13/28 & 2/7 \end{bmatrix}$

/4