Station Higher Derivatives

Exercises

In Exercises 1–16, calculate y'' and y'''.

1. $y = 14x^2$
2. $y = 7 - 2x$
3. $y = x^4 - 25x^2 + 2x$
4. $y = 4t^3 - 9t^2 + 7$
5. $y = \frac{4}{3}\pi r^3$
6. $y = \sqrt{x}$
7. $y = 20t^{4/5} - 6t^{2/3}$
8. $y = x^{-9/5}$
9. $y = z - \frac{4}{z}$
10. $y = 5t^{-3} + 7t^{-8/3}$
11. $y = \theta^2(2\theta + 7)$
12. $y = (x^2 + x)(x^3 + 1)$
13. $y = \frac{x - 4}{x}$
14. $y = \frac{1}{1 - x}$
15. $y = x^5 e^x$
16. $y = \frac{e^x}{x}$

In Exercises 17–26, calculate the derivative indicated.

17. $f^{(4)}(1), \quad f(x) = x^4$
18. $g'''(-1), \quad g(t) = -4t^{-5}$
19. $\left.\frac{d^2y}{dt^2}\right|_{t=1}, \quad y = 4t^{-3} + 3t^2$
20. $\left.\frac{d^4f}{dt^4}\right|_{t=1}, \quad f(t) = 6t^9 - 2t^5$
21. $\left.\frac{d^4x}{dt^4}\right|_{t=16}, \quad x = t^{-3/4}$
22. $f'''(4), \quad f(t) = 2t^2 - t$
23. $f'''(-3), \quad f(x) = 4e^x - x^3$
24. $f''(1), \quad f(t) = \frac{t}{t+1}$
25. $h''(1), \quad h(w) = \sqrt{w}e^w$
26. $g''(0), \quad g(s) = \frac{e^s}{s+1}$

27. Calculate $y^{(k)}(0)$ for $0 \le k \le 5$, where $y = x^4 + ax^3 + bx^2 + cx + d$ (with a, b, c, d the constants).

28. Which of the following satisfy $f^{(k)}(x) = 0$ for all $k \ge 6$?
(a) $f(x) = 7x^4 + 4 + x^{-1}$
(b) $f(x) = x^3 - 2$
(c) $f(x) = \sqrt{x}$
(d) $f(x) = 1 - x^6$
(e) $f(x) = x^{9/5}$
(f) $f(x) = 2x^2 + 3x^5$

29. Use the result in Example 3 to find $\frac{d^6}{dx^6}x^{-1}$.

30. Calculate the first five derivatives of $f(x) = \sqrt{x}$.
(a) Show that $f^{(n)}(x)$ is a multiple of $x^{-n+1/2}$.
(b) Show that $f^{(n)}(x)$ alternates in sign as $(-1)^{n-1}$ for $n \ge 1$.
(c) Find a formula for $f^{(n)}(x)$ for $n \ge 2$. *Hint:* Verify that the coefficient is $\pm 1 \cdot 3 \cdot 5 \cdots \frac{2n-3}{2^n}$.

In Exercises 31–36, find a general formula for $f^{(n)}(x)$.

31. $f(x) = x^{-2}$
32. $f(x) = (x + 2)^{-1}$
33. $f(x) = x^{-1/2}$
34. $f(x) = x^{-3/2}$
35. $f(x) = xe^{-x}$
36. $f(x) = x^2e^x$

37. (a) Find the acceleration at time $t = 5$ min of a helicopter whose height is $s(t) = 300t - 4t^3$ m.
(b) Plot the acceleration $h''(t)$ for $0 \le t \le 6$. How does this graph show that the helicopter is slowing down during this time interval?

38. Find an equation of the tangent to the graph of $y = f'(x)$ at $x = 3$, where $f(x) = x^4$.

39. Figure 5 shows f, f', and f''. Determine which is which.

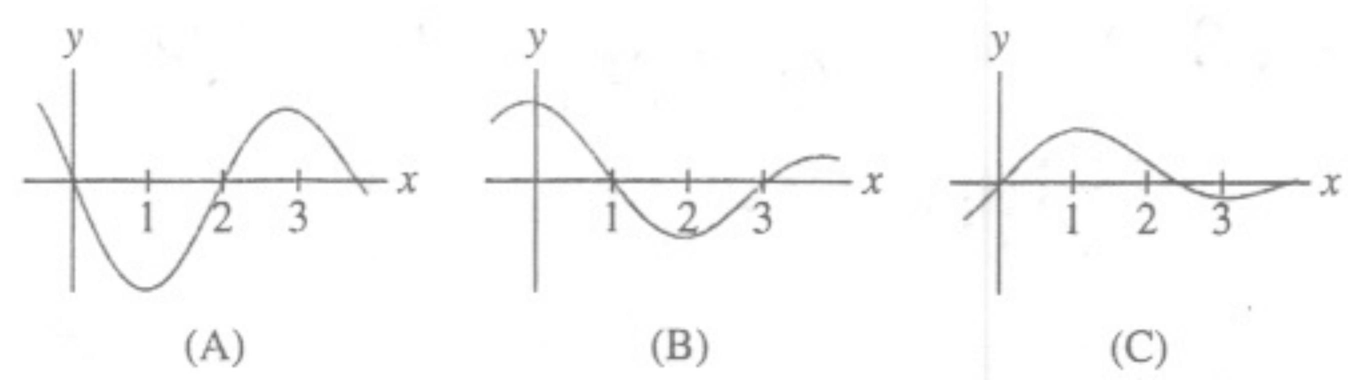


FIGURE 5

40. The second derivative f'' is shown in Figure 6. Which of (A) or (B) is the graph of f and which is f'?

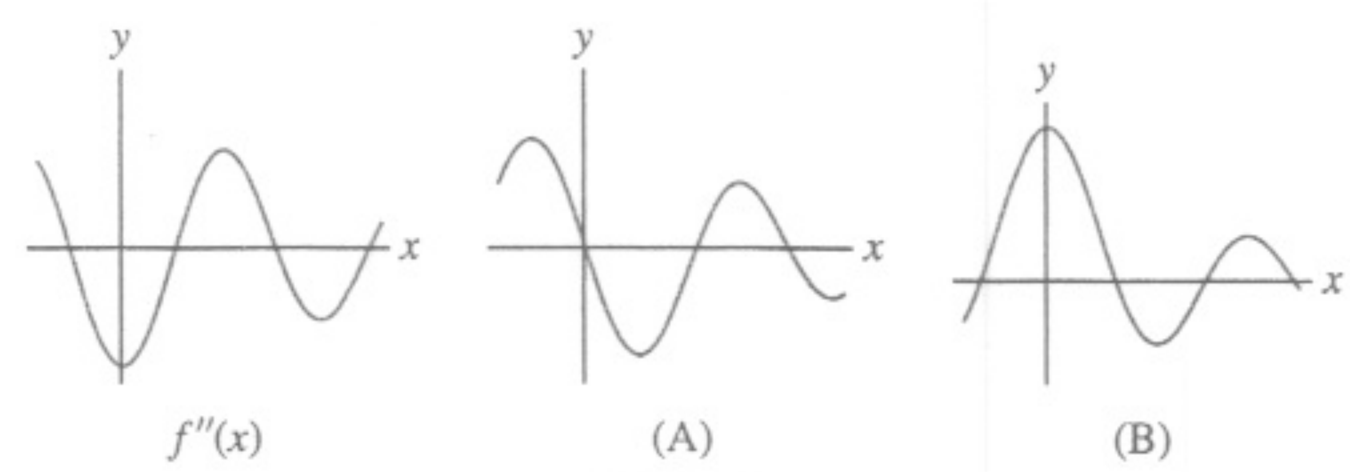


FIGURE 6

41. Figure 7 shows the graph of the position s of an object as a function of time t. Determine the intervals on which the acceleration is positive.

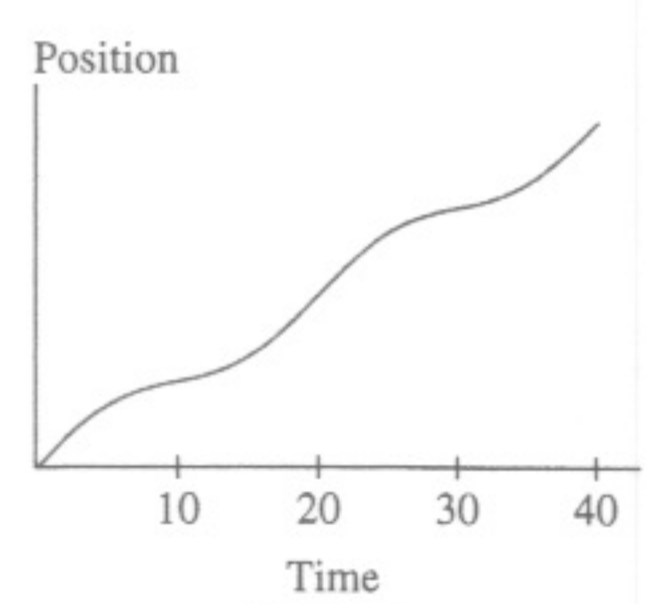


FIGURE 7

42. Find a polynomial $f(x)$ that satisfies the equation $xf''(x) + f(x) = x^2$.

43. Find all values of n such that $y = x^n$ satisfies

$$x^2y'' - 2xy' = 4y$$

5. $d' = 2$ **7.** $dV/dr = 3\pi r^2$

9. **(a)** 100 km/hour **(b)** 100 km/hour **(c)** 0 km/hour **(d)** −50 km/hour

11. **(a)** (i) **(b)** (ii) **(c)** (iii)

13. $\frac{dT}{dt} \approx -1.5625°\text{C/hour}$

15. -8×10^{-6} 1/s

17. $\left.\frac{dT}{dh}\right|_{h=30} \approx 2.94°\text{C/km}$; $\left.\frac{dT}{dh}\right|_{h=70} \approx -3.33°\text{C/km}$; $\frac{dT}{dh} = 0$ over the interval [13, 23], and near the points $h = 50$ and $h = 90$.

19. $v'_{\text{esc}}(r) = -1.41 \times 10^7 r^{-3/2}$

21. $t = \frac{5}{2}$ s

23. The particle passes through the origin when $t = 0$ seconds and when $t = 3\sqrt{2} \approx 4.24$ seconds. The particle is instantaneously motionless when $t = 0$ seconds and when $t = 3$ seconds.

25. Maximum velocity: 200 m/s; maximum height: 2040.82 m

27. Initial velocity: $v_0 = 19.6$ m/s; maximum height: 19.6 m

31. **(a)** $\frac{dV}{dv} = -1$ **(b)** −4

35. Rate of change of BSA with respect to mass: $\frac{\sqrt{5}}{20\sqrt{m}}$; $m = 70$ kg, rate of change is $\approx 0.0133631\,\frac{\text{m}^2}{\text{kg}}$; $m = 80$ kg, rate of change is $\frac{1}{80}\frac{\text{m}^2}{\text{kg}}$; BSA increases more rapidly at lower body mass.

37. 2

39. $\sqrt{2} - \sqrt{1} \approx \frac{1}{2}$; the actual value, to six decimal places, is 0.414214. $\sqrt{101} - \sqrt{100} \approx .05$; the actual value, to six decimal places, is 0.0498756.

41.
- $F(65) = 282.75$ ft
- Increasing speed from 65 to 66 therefore increases stopping distance by approximately 7.6 ft.
- The actual increase in stopping distance when speed increases from 65 mph to 66 mph is $F(66) - F(65) = 290.4 - 282.75 = 7.65$ feet, which differs by less than one percent from the estimate found using the derivative.

43. The cost of producing 2000 bagels is \$796. The cost of the 2001st bagel is approximately \$0.244, which is indistinguishable from the estimated cost.

45. An increase in oil prices of a dollar leads to a decrease in demand of 0.5625 barrels a year, and a decrease of a dollar leads to an *increase* in demand of 0.5625 barrels a year.

47. $\frac{dB}{dI} = \frac{2k}{3I^{1/3}}$; $\frac{dH}{dW} = \frac{3k}{2}W^{1/2}$

(a) As I increases, $\frac{dB}{dI}$ shrinks, so that the rate of change of perceived intensity decreases as the actual intensity increases.

(b) As W increases, $\frac{dH}{dW}$ increases as well, so that the rate of change of perceived weight increases as weight increases.

49. **(a)** The average income among households in the bottom rth part is

$$\frac{F(r)T}{rN} = \frac{F(r)}{r} \cdot \frac{T}{N} = \frac{F(r)}{r}A.$$

(b) The average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\frac{F(r+\Delta r)T - F(r)T}{\Delta r N} = \frac{F(r+\Delta r) - F(r)}{\Delta r} \cdot \frac{T}{N} = \frac{F(r+\Delta r) - F(r)}{\Delta r}A$$

(c) Take the result from part (b) and let $\Delta r \to 0$. Because

$$\lim_{\Delta r \to 0} \frac{F(r+\Delta r) - F(r)}{\Delta r} = F'(r),$$

we find that a household in the $100r$th percentile has income $F'(r)A$.

(d) The point P in Figure 14(B) has an r-coordinate of 0.6, while the point Q has an r-coordinate of roughly 0.75. Thus, on curve L_1, 40% of households have $F'(r) > 1$ and therefore have above-average income. On curve L_2, roughly 25% of households have above-average income.

53. By definition, the slope of the line through (0, 0) and $(x, C(x))$ is

$$\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{\text{avg}}(x).$$

- At point A, average cost is greater than marginal cost.
- At point B, average cost is greater than marginal cost.
- At point C, average cost and marginal cost are nearly the same.
- At point D, average cost is less than marginal cost.

Section 3.5 Preliminary Questions

1. The first derivative of stock prices must be positive, while the second derivative must be negative.

2. True

3. All quadratic polynomials

4. e^x

Section 3.5 Exercises

1. $y'' = 28$ and $y''' = 0$

3. $y'' = 12x^2 - 50$ and $y''' = 24x$

5. $y'' = 8\pi r$ and $y''' = 8\pi$

7. $y'' = -\frac{16}{5}t^{-6/5} + \frac{4}{3}t^{-4/3}$ and $y''' = \frac{96}{25}t^{-11/5} - \frac{16}{9}t^{-7/3}$

9. $y'' = -8z^{-3}$ and $y''' = 24z^{-4}$

11. $y'' = 12\theta + 14$ and $y''' = 12$

13. $y'' = -8x^{-3}$ and $y''' = 24x^{-4}$

15. $y'' = (x^5 + 10x^4 + 20x^3)e^x$ and $y''' = (x^5 + 15x^4 + 60x^3 + 60x^2)e^x$

17. $f^{(4)}(1) = 24$ **19.** $\left.\frac{d^2y}{dt^2}\right|_{t=1} = 54$

21. $\left.\frac{d^4x}{dt^4}\right|_{t=16} = \frac{3465}{134217728}$ **23.** $f'''(-3) = 4e^{-3} - 6$

25. $h''(1) = \frac{7}{4}e$

27. $y^{(0)}(0) = d$, $y^{(1)}(0) = c$, $y^{(2)}(0) = 2b$, $y^{(3)}(0) = 6a$, $y^{(4)}(0) = 24$, and $y^{(5)}(0) = 0$

29. $\frac{d^6}{dx^6}x^{-1} = 720x^{-7}$

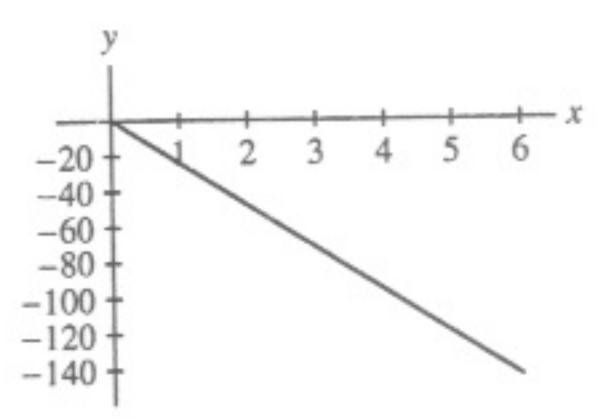
31. $f^{(n)}(x) = (-1)^n (n+1)! x^{-(n+2)}$

33. $f^{(n)}(x) = (-1)^n \frac{(2n-1)\times(2n-3)\times\ldots\times 1}{2^n} x^{-(2n+1)/2}$

35. $f^{(n)}(x) = (-1)^n (x-n)e^{-x}$

37. **(a)** $a(5) = -120 \text{ m/min}^2$

(b) The acceleration of the helicopter for $0 \le t \le 6$ is shown in the figure below. As the acceleration of the helicopter is negative, the velocity of the helicopter must be decreasing. Because the velocity is positive for $0 \le t \le 6$, the helicopter is slowing down.



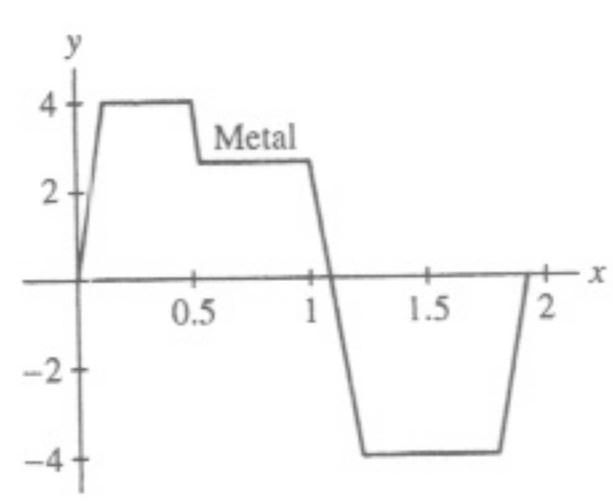
39. (a) f'' (b) f' (c) f

41. Roughly from time 10 to time 20 and from time 30 to time 40

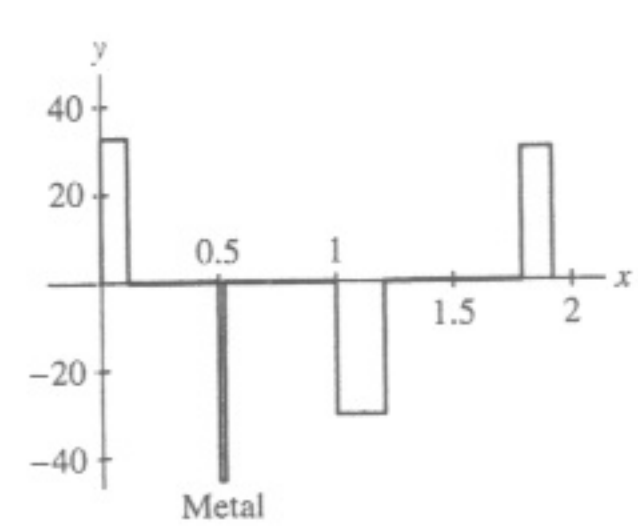
43. $n = -2$ and $n = 3$

45. **(a)** $v(t) = -5.12$ m/s **(b)** $v(t) = -7.25$ m/s

47. A possible plot of the drill bit's vertical velocity follows:



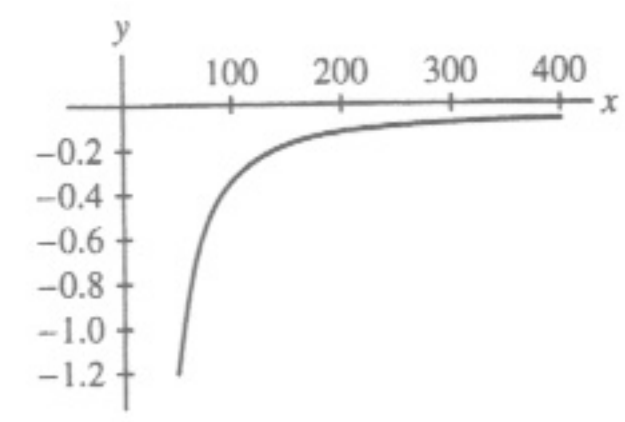
A graph of the acceleration is extracted from this graph:



49. **(a)** Traffic speed must be reduced when the road gets more crowded so we expect $\frac{dS}{dQ}$ to be negative.

(b) The decrease in speed due to a one-unit increase in density is approximately $\frac{dS}{dQ}$ (a negative number). Since $\frac{d^2S}{dQ^2} = 5764Q^{-3} > 0$ is positive, this tells us that $\frac{dS}{dQ}$ gets larger as Q increases.

(c) dS/dQ is plotted below. The fact that this graph is increasing shows that $d^2S/dQ^2 > 0$.



51.

$$f'(x) = -\frac{3}{(x-1)^2} = (-1)^1 \frac{3\cdot 1}{(x-1)^{1+1}};$$
$$f''(x) = \frac{6}{(x-1)^3} = (-1)^2 \frac{3\cdot 2\cdot 1}{(x-1)^{2+1}};$$
$$f'''(x) = -\frac{18}{(x-1)^4} = (-1)^3 \frac{3\cdot 3!}{(x-1)^{3+1}}; \text{ and}$$
$$f^{(4)}(x) = \frac{72}{(x-1)^5} = (-1)^4 \frac{3\cdot 4!}{(x-1)^{4+1}}.$$

From the pattern observed above, we conjecture

$$f^{(k)}(x) = (-1)^k \frac{3\cdot k!}{(x-1)^{k+1}}.$$

53. 99!

55. $(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$;
$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$

Section 3.6 Preliminary Questions

1. **(a)** $\frac{d}{dx}(\sin x + \cos x) = -\sin x + \cos x$

(b) $\frac{d}{dx} \sec x = \sec x \tan x$

(c) $\frac{d}{dx} \cot x = -\csc^2 x$

2. **(a)** This function can be differentiated using the Product Rule.

(b) We have not yet discussed how to differentiate a function like this.

(c) This function can be differentiated using the Product Rule.

3. 0

4. The difference quotient for the function $\sin x$ involves the expression $\sin(x+h)$. The addition formula for the sine function is used to expand this expression as
$\sin(x+h) = \sin x \cos h + \sin h \cos x$.

Section 3.6 Exercises

1. $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)$ **3.** $y = 2x + 1 - \frac{\pi}{2}$

5. $f'(x) = -\sin^2 x + \cos^2 x$ **7.** $f'(x) = 2\sin x \cos x$

9. $H'(t) = 2\sin t \sec^2 t \tan t + \sec t$

11. $f'(\theta) = \left(\tan^2\theta + \sec^2\theta\right)\sec\theta$

13. $f'(x) = (2x^4 - 4x^{-1})\sec x \tan x + \sec x(8x^3 + 4x^{-2})$

15. $y' = \frac{\theta \sec\theta \tan\theta - \sec\theta}{\theta^2}$ **17.** $R'(y) = \frac{4\cos y - 3}{\sin^2 y}$