2.5 SUMMARY

- When $f(x)$ is known to be continuous at $x = c$, the limit can be evaluated by substitution: $\lim_{x \to c} f(x) = f(c)$.
- We say that $f(x)$ is *indeterminate* (or has an *indeterminate form*) at $x = c$ if the formula for $f(c)$ yields an undefined expression of the type

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$$

- If $f(x)$ is indeterminate at $x = c$, try to transform $f(x)$ algebraically into a new expression that is defined and continuous at $x = c$. Then evaluate by substitution.

2.5 EXERCISES

Preliminary Questions

1. Which of the following is indeterminate at $x = 1$?

$$\frac{x^2+1}{x-1}, \quad \frac{x^2-1}{x+2}, \quad \frac{x^2-1}{\sqrt{x+3}-2}, \quad \frac{x^2+1}{\sqrt{x+3}-2}$$

2. Give counterexamples to show that these statements are false:

(a) If $f(c)$ is indeterminate, then the right- and left-hand limits as $x \to c$ are not equal.

(b) If $\lim_{x \to c} f(x)$ exists, then $f(c)$ is not indeterminate.

(c) If $f(x)$ is undefined at $x = c$, then $f(x)$ has an indeterminate form at $x = c$.

3. The method for evaluating limits discussed in this section is sometimes called "simplify and plug in." Explain how it actually relies on the property of continuity.

Exercises

In Exercises 1–4, show that the limit leads to an indeterminate form. Then carry out the two-step procedure: Transform the function algebraically and evaluate using continuity.

1. $\lim_{x \to 6} \frac{x^2 - 36}{x - 6}$

2. $\lim_{h \to 3} \frac{9 - h^2}{h - 3}$

3. $\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$

4. $\lim_{t \to 9} \frac{2t - 18}{5t - 45}$

In Exercises 5–34, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

5. $\lim_{x \to 7} \frac{x - 7}{x^2 - 49}$

6. $\lim_{x \to 8} \frac{x^2 - 64}{x - 9}$

7. $\lim_{x \to -2} \frac{x^2 + 3x + 2}{x + 2}$

8. $\lim_{x \to 8} \frac{x^3 - 64x}{x - 8}$

9. $\lim_{x \to 5} \frac{2x^2 - 9x - 5}{x^2 - 25}$

10. $\lim_{h \to 0} \frac{(1 + h)^3 - 1}{h}$

11. $\lim_{x \to -\frac{1}{2}} \frac{2x + 1}{2x^2 + 3x + 1}$

12. $\lim_{x \to 3} \frac{x^2 - x}{x^2 - 9}$

13. $\lim_{x \to 2} \frac{3x^2 - 4x - 4}{2x^2 - 8}$

14. $\lim_{h \to 0} \frac{(3 + h)^3 - 27}{h}$

15. $\lim_{t \to 0} \frac{4^{2t} - 1}{4^t - 1}$

16. $\lim_{h \to 4} \frac{(h + 2)^2 - 9h}{h - 4}$

17. $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}$

18. $\lim_{t \to -2} \frac{2t + 4}{12 - 3t^2}$

19. $\lim_{y \to 3} \frac{y^2 + y - 12}{y^3 - 10y + 3}$

20. $\lim_{h \to 0} \frac{\frac{1}{(h + 2)^2} - \frac{1}{4}}{h}$

21. $\lim_{h \to 0} \frac{\sqrt{2 + h} - 2}{h}$

22. $\lim_{x \to 8} \frac{\sqrt{x - 4} - 2}{x - 8}$

23. $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - \sqrt{8 - x}}$

24. $\lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}}$

25. $\lim_{x \to 4} \left(\frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right)$

26. $\lim_{x \to 0+} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2 + x}} \right)$

27. $\lim_{x \to 0} \frac{\cot x}{\csc x}$

28. $\lim_{\theta \to \frac{\pi}{2}} \frac{\cot \theta}{\csc \theta}$

29. $\lim_{t \to 2} \frac{2^{2t} + 2^t - 20}{2^t - 4}$

30. $\lim_{x \to 1} \left(\frac{1}{1 - x} - \frac{2}{1 - x^2} \right)$

31. $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{\tan x - 1}$

32. $\lim_{\theta \to \frac{\pi}{2}} (\sec \theta - \tan \theta)$

33. $\lim_{\theta \to \frac{\pi}{4}} \left(\frac{1}{\tan \theta - 1} - \frac{2}{\tan^2 \theta - 1} \right)$

34. $\lim_{x \to \frac{\pi}{3}} \frac{2\cos^2 x + 3\cos x - 2}{2\cos x - 1}$

35. GU Use a plot of $f(x) = \dfrac{x-4}{\sqrt{x}-\sqrt{8-x}}$ to estimate $\lim_{x\to 4} f(x)$ to two decimal places. Compare with the answer obtained algebraically in Exercise 23.

36. GU Use a plot of $f(x) = \dfrac{1}{\sqrt{x}-2} - \dfrac{4}{x-4}$ to estimate $\lim_{x\to 4} f(x)$ numerically. Compare with the answer obtained algebraically in Exercise 25.

In Exercises 37–42, evaluate using the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

37. $\lim_{x\to 2} \dfrac{x^3-8}{x-2}$

38. $\lim_{x\to 3} \dfrac{x^3-27}{x^2-9}$

39. $\lim_{x\to 1} \dfrac{x^2-5x+4}{x^3-1}$

40. $\lim_{x\to -2} \dfrac{x^3+8}{x^2+6x+8}$

41. $\lim_{x\to 1} \dfrac{x^4-1}{x^3-1}$

42. $\lim_{x\to 27} \dfrac{x-27}{x^{1/3}-3}$

43. Evaluate $\lim_{h\to 0} \dfrac{\sqrt[4]{1+h}-1}{h}$. *Hint:* Set $x = \sqrt[4]{1+h}$ and rewrite as a limit as $x \to 1$.

44. Evaluate $\lim_{h\to 0} \dfrac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}$. *Hint:* Set $x = \sqrt[6]{1+h}$ and rewrite as a limit as $x \to 1$.

In Exercises 45–54, evaluate in terms of the constant a.

45. $\lim_{x\to 0} (2a+x)$

46. $\lim_{h\to -2} (4ah+7a)$

47. $\lim_{t\to -1} (4t-2at+3a)$

48. $\lim_{h\to 0} \dfrac{(3a+h)^2-9a^2}{h}$

49. $\lim_{h\to 0} \dfrac{2(a+h)^2-2a^2}{h}$

50. $\lim_{x\to a} \dfrac{(x+a)^2-4x^2}{x-a}$

51. $\lim_{x\to a} \dfrac{\sqrt{x}-\sqrt{a}}{x-a}$

52. $\lim_{h\to 0} \dfrac{\sqrt{a+2h}-\sqrt{a}}{h}$

53. $\lim_{x\to 0} \dfrac{(x+a)^3-a^3}{x}$

54. $\lim_{h\to a} \dfrac{\frac{1}{h}-\frac{1}{a}}{h-a}$

Further Insights and Challenges

In Exercises 55–58, find all values of c such that the limit exists.

55. $\lim_{x\to c} \dfrac{x^2-5x-6}{x-c}$

56. $\lim_{x\to 1} \dfrac{x^2+3x+c}{x-1}$

57. $\lim_{x\to 1} \left(\dfrac{1}{x-1} - \dfrac{c}{x^3-1}\right)$

58. $\lim_{x\to 0} \dfrac{1+cx^2-\sqrt{1+x^2}}{x^4}$

59. For which sign $\pm$ does the following limit exist?

$$\lim_{x\to 0} \left(\frac{1}{x} \pm \frac{1}{x(x-1)}\right)$$

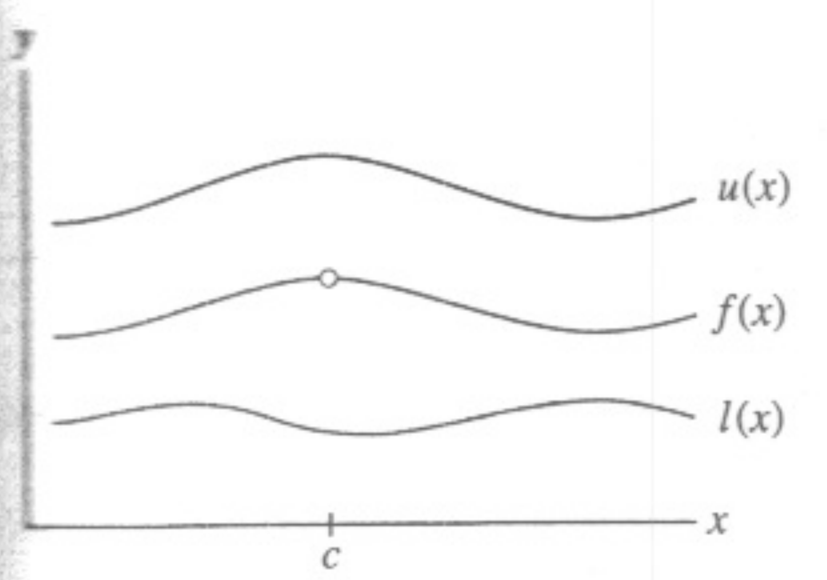


FIGURE 1 $f(x)$ is trapped between $l(x)$ and $u(x)$ (but not squeezed at $x = c$).

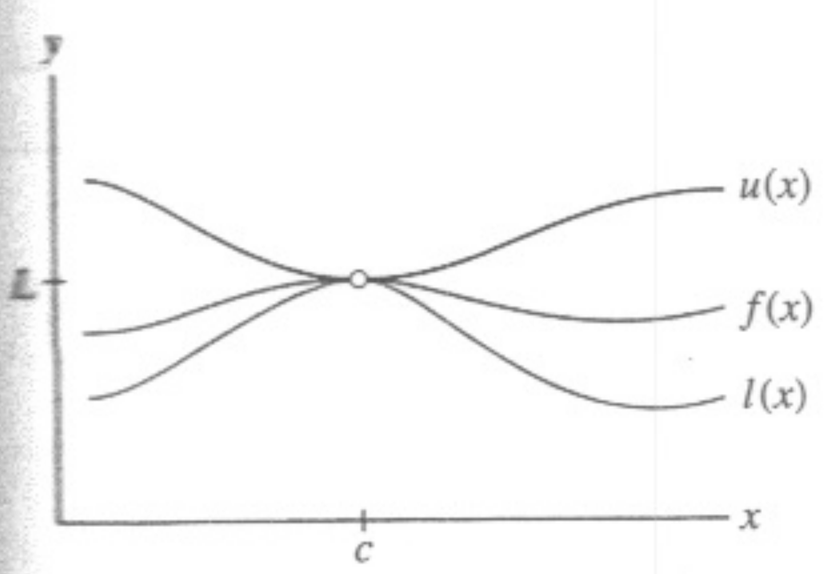


FIGURE 2 $f(x)$ is squeezed by $l(x)$ and $u(x)$ at $x = c$.

2.6 Trigonometric Limits

In our study of the derivative, we will need to evaluate certain limits involving transcendental functions such as sine and cosine. The algebraic techniques of the previous section are often ineffective for such functions, and other tools are required. In this section, we discuss one such tool—the Squeeze Theorem—and use it to evaluate the trigonometric limits needed in Section 3.6.

The Squeeze Theorem

Consider a function $f(x)$ that is "trapped" between two functions $l(x)$ and $u(x)$ on an interval I. In other words,

$$l(x) \le f(x) \le u(x) \quad \text{for all } x \in I$$

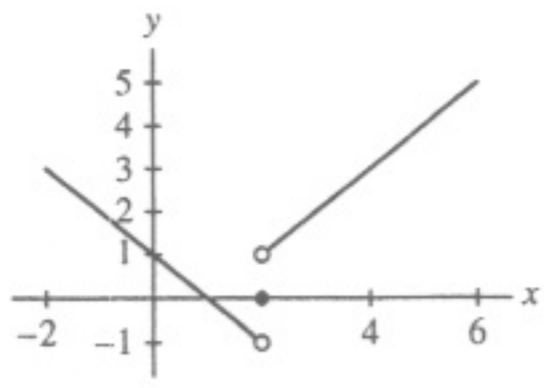
Thus, the graph of $f(x)$ lies between the graphs of $l(x)$ and $u(x)$ (Figure 1).

The Squeeze Theorem applies when $f(x)$ is not just trapped but **squeezed** at a point $x = c$ (Figure 2). By this we mean that for all $x \ne c$ in some open interval containing c,

$$l(x) \le f(x) \le u(x) \qquad \text{and} \qquad \lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$$

We do not require that $f(x)$ be defined at $x = c$, but it is clear graphically that $f(x)$ must approach the limit L, as stated in the next theorem. See Appendix D for a proof.

53. The function f is neither left- nor right-continuous at $x = 2$.

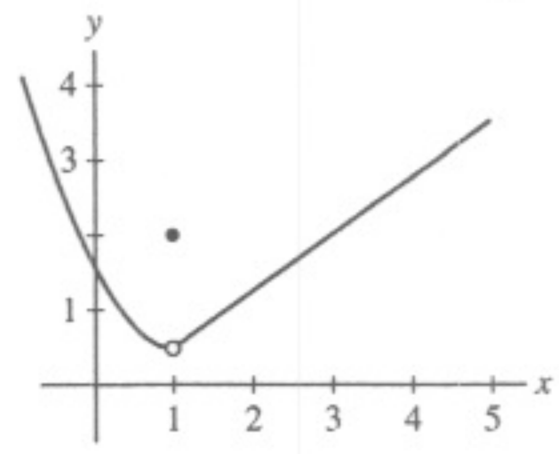


55. $\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4}(x+4) = 8 \neq 10 = f(4)$

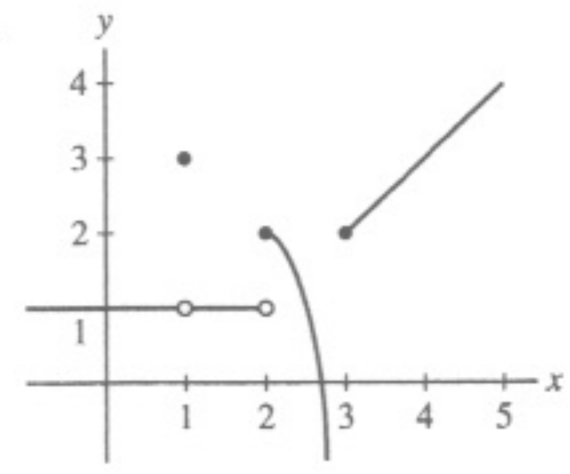
57. $c = \frac{5}{3}$ **59.** $a = 2$ and $b = 1$

61. **(a)** No **(b)** $g(1) = -\frac{\pi}{2}$

63.



65.

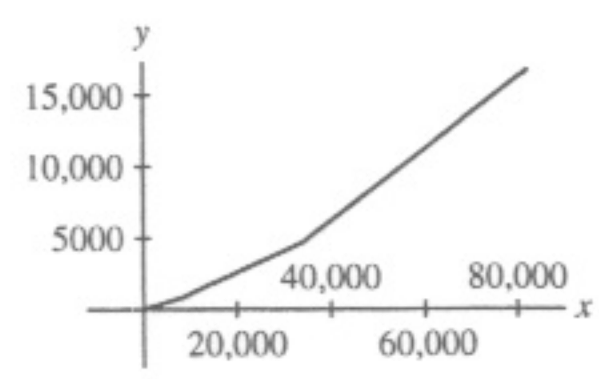


67. -6 **69.** $\frac{1}{3}$ **71.** -1 **73.** $\frac{1}{32}$ **75.** 27 **77.** 1000 **79.** $\frac{\pi}{2}$

81. No. Take $f(x) = -x^{-1}$ and $g(x) = x^{-1}$

83. $f(x) = |g(x)|$ is a composition of the continuous functions $g(x)$ and $|x|$

85. No.



87. $f(x) = 3$ and $g(x) = [x]$

Section 2.5 Preliminary Questions

1. $\frac{x^2-1}{\sqrt{x+3}-2}$

2. **(a)** $f(x) = \frac{x^2-1}{x-1}$ **(b)** $f(x) = \frac{x^2-1}{x-1}$ **(c)** $f(x) = \frac{1}{x}$

3. The "simplify and plug-in" strategy is based on simplifying a function which is indeterminate to a continuous function. Once the simplification has been made, the limit of the remaining continuous function is obtained by evaluation.

Section 2.5 Exercises

1. $\lim_{x\to 6} \frac{x^2-36}{x-6} = \lim_{x\to 6} \frac{(x-6)(x+6)}{x-6} = \lim_{x\to 6}(x+6) = 12$

3. 0 **5.** $\frac{1}{14}$ **7.** -1 **9.** $\frac{11}{10}$ **11.** 2 **13.** 1 **15.** 2 **17.** $\frac{1}{8}$

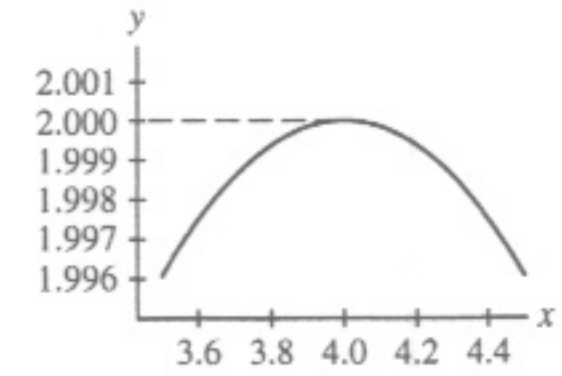
19. $\frac{7}{17}$

21. Limit does not exist.

- As $h \to 0+$, $\frac{\sqrt{h+2}-2}{h} \to -\infty$.
- As $h \to 0-$, $\frac{\sqrt{h+2}-2}{h} \to \infty$.

23. 2 **25.** $\frac{1}{4}$ **27.** 1 **29.** 9 **31.** $\frac{\sqrt{2}}{2}$ **33.** $\frac{1}{2}$

35. $\lim_{x\to 4} f(x) \approx 2.00$; to two decimal places, this matches the [illegible] of 2 obtained in Exercise 23.



37. 12 **39.** -1 **41.** $\frac{4}{3}$ **43.** $\frac{1}{4}$ **45.** $2a$ **47.** $-4+5a$ **49.** [illegible]

51. $\frac{1}{2\sqrt{a}}$ **53.** $3a^2$ **55.** $c = -1$ and $c = 6$ **57.** $c = 3$ **59.** [illegible]

Section 2.6 Preliminary Questions

1. $\lim_{x\to 0} f(x) = 0$; No

2. Assume that for $x \neq c$ (in some open interval containing c),

$$l(x) \le f(x) \le u(x)$$

and that $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. Then $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = L.$$

3. **(a)**

Section 2.6 Exercises

1. For all $x \neq 1$ on the open interval $(0, 2)$ containing $x = 1$, $\ell(x) \le f(x) \le u(x)$. Moreover,

$$\lim_{x\to 1} \ell(x) = \lim_{x\to 1} u(x) = 2.$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 1} f(x) = 2.$$

3. $\lim_{x\to 7} f(x) = 6$

5. **(a)** *not* sufficient information **(b)** $\lim_{x\to 1} f(x) = 1$ **(c)** $\lim_{x\to 1} f(x) = 3$

7. $\lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$ **9.** $\lim_{x\to 1}(x-1)\sin\frac{\pi}{x-1} = 0$

11. $\lim_{t\to 0}(2^t - 1)\cos\frac{1}{t} = 0$

13. $\lim_{t\to 2}(t^2-4)\cos\frac{1}{t-2} = 0$

15. $\lim_{\theta\to\frac{\pi}{2}} \cos\theta\cos(\tan\theta) = 0$

17. 1 **19.** 3 **21.** 1 **23.** 0 **25.** $\frac{2\sqrt{2}}{\pi}$ **27.** **(b)** $L = 14$ **29.** [illegible]

31. $\frac{1}{5}$ **33.** $\frac{7}{3}$ **35.** $\frac{1}{25}$ **37.** 6 **39.** $-\frac{3}{4}$ **41.** $\frac{1}{2}$ **43.** $\frac{6}{5}$ **45.** [illegible]

47. 0 **49.** -1 **53.** $-\frac{9}{2}$

55. $\lim_{t\to 0+} \frac{\sqrt{1-\cos t}}{t} = \frac{\sqrt{2}}{2}$; $\lim_{t\to 0-} \frac{\sqrt{1-\cos t}}{t} = -\frac{\sqrt{2}}{2}$

59. **(a)**

x	$c - .01$	$c - .001$	$c + .001$	$c +$ [illegible]
$\frac{\sin x - \sin c}{x - c}$	.999983	.99999983	.99999983	.9999[illegible]

Here $c = 0$ and $\cos c = 1$.