Worksheet 3: Solutions and Teacher Notes

1. First, set the derivative equal to zero to find critical numbers. That is, solve the equation $f'(x) = 4x^3 - 12x^2 = 0$ to obtain the critical numbers $x = 0$ and $x = 3$. A first derivative test reveals that the minimum must occur at $x = 3$.

	$x < 0$	$0 < x < 3$	$x > 3$
$f'(x)$	negative	negative	positive
$f(x)$	decreasing	decreasing	increasing

Substituting $x = 3$ into the original function and setting the result equal to 7 yields $k = 34$. That is, the equation $f(3) = 3^4 - 4 \cdot 3^3 + k = 7$ has solution $k = 34$.

2. a. Since we are looking for the minimum of P, we compute $P'(t) = \dfrac{t^2 - 1}{(t^2+1)^2}$ and solve $P'(t) = 0$ to obtain $t = 1$ and $t = -1$. Since $t = -1$ is not in our domain, our only critical point is $t = 1$. Since $t = 1$ is the only critical point for $t \geq 0$ and is negative for $0 \leq t < 1$ and positive for $t > 1$, we have established that the minimum proportion of the normal oxygen level in the pond occurs at $t = 1$ week.

   b. Since we are looking for the maximum of $P'(t)$, we need to compute the derivative of $P'(t)$, $P''(t) = \dfrac{-2t(t^2-3)}{(t^2+1)^3} = 0$. We then solve $P''(t) = 0$ to obtain $t = 0$ and $t = \pm\sqrt{3}$. Since $t = \sqrt{3}$ is the only critical point in the interior of our domain and $P''(t)$ changes from positive to negative at $t = \sqrt{3}$ (which means $P'(t)$ goes from increasing to decreasing), then the proportion of the normal oxygen level in the pond is increasing most rapidly at $t = \sqrt{3}$.

3. The key to understanding this problem is in realizing that the tangent line to the curve is horizontal, which means that the slope, $\dfrac{dy}{dx}$, must equal zero. Note that $\dfrac{dy}{dx} = \dfrac{3-x}{y+2} = 0$ only at $x = 3$, $y \neq -2$. We conclude that the line $y = -3$ is tangent to the curve at the point $(3, -3)$. To determine the type of point we are working with, we must use the second derivative test. Using the quotient rule to find $\dfrac{d^2y}{dx^2} = \dfrac{-(y+2) - (3-x)\dfrac{dy}{dx}}{(y+2)^2}$ and evaluating at the point $(3, -3)$, we find the value of the second derivative to be positive $\left(\dfrac{d^2y}{dx^2} = 1\right)$, which means the point $(3, -3)$ must be a relative (local) minimum point.

4. a. By examining the graph of R on the graphing calculator, you can see that $R(t)$ is positive for $0 < t < 1.572$, negative for $1.572 < t < 3.514$, and positive for $3.514 < t < 4$. This means that the balloon rises, falls and then rises. The maximum altitude is either at $t = 1.572$ or $t = 4$. Since the area below the x-axis for $1.572 < t < 3.514$ is larger than the area above the x-axis for $3.514 < t < 4$, the decrease in altitude is greater than the increase in altitude during the last time interval $(3.514 < t < 4)$, so the maximum altitude occurs at $t = 1.572$.

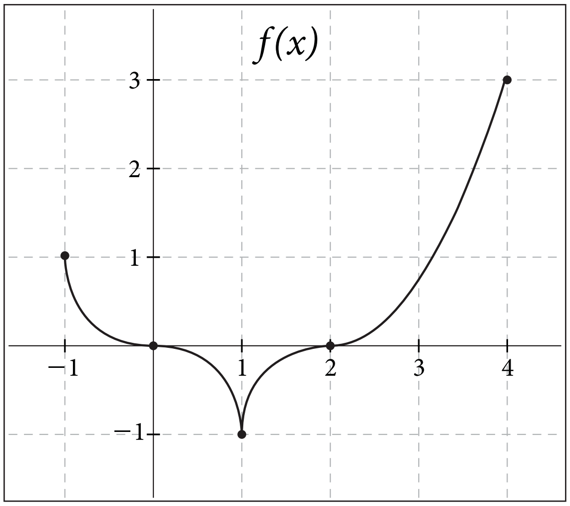
   Indeed, we can check that the altitude at $t = 1.572$ hours is $\int_0^{1.572} R(t)\,dt = 5.779$ kilometers, while the altitude at $t = 4$ hours is $\int_0^4 R(t)\,dt = 2.667$ km.

   b. We are looking for the time when $R(t)$ is a maximum. Therefore, we need to set the derivative of $R(t)$, $R'(t)$, equal to zero: $R'(t) = 3t^2 - 8t = 0$ when $t = 0$ or $t = 8/3$. Since $t = 8/3$ is the only critical point in the interior of our domain and $R''(t) = 6t$ is positive at $t = 8/3$, R has a local minimum there. (In fact, $R(8/3) = -3.481$ km/hr, showing that the altitude of the balloon is decreasing at $t = 8/3$.) Therefore, we should check the values of $R(t)$ at the two endpoints $t = 0$ and $t = 4$. Since $R(0) = R(4) = 6$ kilometers per hour, the altitude of the balloon is increasing most rapidly at $t = 0$ and $t = 4$.

   I like to include problems like this because students always assume the maximum or minimum will occur at the critical point they have found (and many textbooks reinforce this idea because that's how ALL of the problems turn out). It's good to occasionally have problems with endpoint answers.

5. a. f has a relative minimum at $x = 1$ since the sign of $f'(x)$ changes from negative to positive at that point, which indicates that the function changes from decreasing to increasing at $x = 1$, giving us a relative minimum.

   b. Since there are no points where $f'(x)$ changes from positive to negative there are no interior candidates for our maximum value. Therefore, the maximum must occur at an endpoint. Since $f(4) = 3 > 1 = f(-1)$, the maximum value of the function is $f(4) = 3$.

c. 

d. Using the Fundamental Theorem of Calculus, $h'(x) = f(x)$. Since $h'(x) = f(x)$ changes from positive to negative at $x = 0$, there is a relative maximum at $x = 0$. Since $h'(x) = f(x)$ changes from negative to positive at $x = 2$, there is relative minimum at $x = 2$.

6. a. $x(t) = x(0) + \int_0^t v(u)\,du = -1 + \int_0^t u^{1/2} \cos u \; du$. Integrating the velocity gives the net change in position and, if we add the initial position to the change in position, we get the new position.

b. We compare values of the position function x at the endpoints and at all critical points. Critical points for the position function occur when the derivative of the position, which is the velocity, equals zero. Since $x'(t) = v(t) = t^{1/2} \cos t = 0$ for $t = 0$, $t = \pi/2$, and $t = 3\pi/2$, the critical points are $t = 0$, $t = \pi/2$ and $t = 3\pi/2$. We use our solution from part a to find the position $x(t)$ at the critical points and endpoints.

t	0	$\pi/2$	$3\pi/2$	6.5
$x(t)$	-1	-0.296	-3.819	-0.887

The particle is farthest from the origin at $t = 3\pi/2$.

c. To determine if velocity is increasing or decreasing, we should check the sign of $v'(t)$. Using the NDeriv feature of the calculator, $v'\left(\frac{3\pi}{2}\right) = 2.171$, which indicates velocity is increasing because its derivative is positive.

This part of the problem has very little to do with extrema, but I try to find every opportunity I can to reinforce students' calculator skills. They seem to get lots of practice evaluating definite integrals on the calculator but much less practice evaluating derivatives.