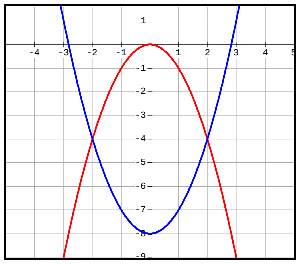
7.1 AREA: set up the integral to find the area bounded by the following curves.

1. With respect to x: $y = -x^2$, $y = x^2 - 8$



Step 1: Put the two equations in $y = f(x)$ form:

These two equations are already in this form, so we are good to go.

Step 2: Find the x-values of the points of intersection:

Subtract the function on the bottom from the function on the top:

$$-x^2 - (x^2 - 8) = 0$$

$$-2x^2 + 8 = 0$$ note: this is the difference of the curves

$$x^2 - 4 = (x+2)(x-2) = 0$$

$$x = \{-2, 2\}$$

Step 3: Integrate the difference of the curves:

Use a definite integral with limits equal to the points of intersection found in Step 2:

$$\int_{-2}^{2} (-2x^2 + 8)\, dx$$

A couple of notes about this process (using the Absolute Value Trick)

- In this problem, we are able to see what the curves look like. If we cannot see them, and we are not sure which curve is on top, we can subtract them in any order and take the absolute value of the result. For example, in this problem, if we subtracted the first curve from the second, we could calculate the area as:

$$\left| \int_{-2}^{2} [(x^2 - 8) - (-x^2)]\, dx \right| = \left| \int_{-2}^{2} (2x^2 - 8)\, dx \right|$$

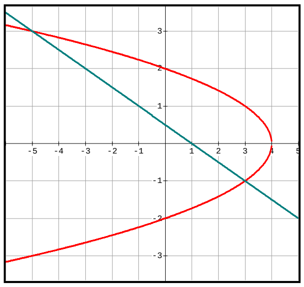
  This will give us the same solution as the one we obtained above because:

$$\left| \int_{-2}^{2} (2x^2 - 8)\, dx \right| = \int_{-2}^{2} (-2x^2 + 8)\, dx$$

- If there are multiple regions between the two curves, you can use the Absolute Value Trick on each individual region and add up all of the resulting positive areas to obtain the total area for the regions.

- Remember that when we are measuring the areas of regions between two curves, **all areas are positive**. That's why the Absolute Value Trick works.

2. With respect to y: $y^2 = 4 - x$, $x + 2y - 1 = 0$



Step 1: Put the two equations in $x = f(y)$ form:

Since we are instructed to perform the integration in terms of y, we want to put each equation in the form $x = f(y)$ to prepare for Step 2:

$y^2 = 4 - x$ $\qquad$ $x + 2y - 1 = 0$

$x = 4 - y^2$ $\qquad$ $x = -2y + 1$

Step 2: Find the y-values of the points of intersection:

Subtract the function on the left from the function on the right:

$4 - y^2 - (-2y + 1) = 0$

$-y^2 + 2y + 3 = 0$ note: this is the difference of the curves

$y^2 - 2y - 3 = (y + 1)(y - 3) = 0$

$y = \{-1, 3\}$

Step 3: Integrate the difference of the curves:

Use a definite integral with limits equal to the points of intersection found in Step 2:

$$\int_{-1}^{3} (-y^2 + 2y + 3)\, dy$$

A thought about functions of the form $x = f(y)$ (using the Old Switcheroo Trick)

Some students have trouble thinking about functions of the form $x = f(y)$ because most of our work is done with forms $y = f(x)$. The student should get used to working with either form. However, in a pinch, you can use the Old Switcheroo Trick. Put simply, it does not matter what the variables are; so you can switch them whenever you like.

For example, in this problem, you could switch the x and y variables and get the same solution, although it would look a little different in integral form. If we switch the variables in the initial equations, we would obtain the following revised initial equations:

$x^2 = 4 - y$ $\qquad$ $y + 2x - 1 = 0$

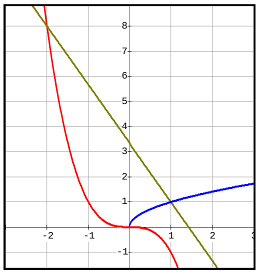
Following all of the above work, we would arrive at a final solution looking like this:

$$\int_{-1}^{3} (-x^2 + 2x + 3)\, dx$$

At this point, we could either switch the variables back to what they were in the original problem, or develop the numerical solution. Note that the variable in the integral is irrelevant:

$$\int_{-1}^{3} (-y^2 + 2y + 3)\, dy \;=\; \int_{-1}^{3} (-x^2 + 2x + 3)\, dx$$

3. $y + x^3 = 0,\quad y = \sqrt{x},\quad 3y + 7x - 10 = 0$



Step 1: Put the three equations in $y = f(x)$ form:

We can use either the form $y = f(x)$ or the form $x = f(y)$ in this problem. We will use the form $y = f(x)$ to prepare for Step 2:

$y + x^3 = 0$	$y = \sqrt{x}$	$3y + 7x - 10 = 0$
$y = -x^3$	$y = \sqrt{x}$	$y = (-\frac{7}{3}x + \frac{10}{3})$

Step 2: Find the y-values of the points of intersection:

Left point of intersection. Subtract the function on the bottom from the function on the top:

$$(-\tfrac{7}{3}x + \tfrac{10}{3}) - (-x^3) = 0$$

$$x^3 - \tfrac{7}{3}x + \tfrac{10}{3} = 0$$ note: this is the difference of the curves

A look at the graph suggests that the intersection occurs at $\boldsymbol{x = -2}$. Substituting $\boldsymbol{x = -2}$ into the equation above confirms this. Alternatively, use a calculator to determine that the intersection occurs at $\boldsymbol{x = -2}$.

Center point of intersection. Note that the "bottom" curve in the graph changes at the center point of intersection. Therefore, we must will break our solution into two integrals, one to the left of the center point of intersection, and one to its right. To find the point of intersection, equate the red and blue curves to find the point of intersection:

$$-x^3 = \sqrt{x}$$

A look at the graph suggests that the intersection occurs at $\boldsymbol{x = 0}$. Substituting $\boldsymbol{x = 0}$ into the equation above confirms this.

Right point of intersection. Subtract the function on the bottom from the function on the top:

$$(-\tfrac{7}{3}x + \tfrac{10}{3}) - \sqrt{x} = 0$$

$$(-\tfrac{7}{3}x + \tfrac{10}{3}) - \sqrt{x} = 0$$ note: this is the difference of the curves

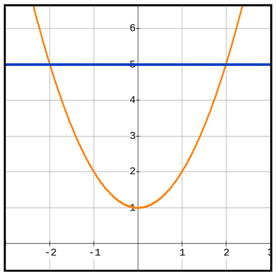
A look at the graph suggests that the intersection occurs at $\boldsymbol{x = 1}$. Substituting $\boldsymbol{x = 1}$ into the equation above confirms this. Alternatively, use a calculator to determine that the intersection occurs at $\boldsymbol{x = 1}$.

Step 3: Integrate the difference of the curves:

In this problem, we need two definite integrals, one on each side of the point where the red and blue curves intersect:

$$\int_{-2}^{0} \left(x^3 - \frac{7}{3}x + \frac{10}{3}\right) dx + \int_{0}^{1} \left(-\frac{7}{3}x + \frac{10}{3} - \sqrt{x}\right) dx$$

4. $y = x^2 + 1, \quad y = 5$



Step 1: Put the two equations in $y = f(x)$ form:

These two equations are already in this form, so we are good to go.

Step 2: Find the x-values of the points of intersection:

Subtract the function on the bottom from the function on the top:

$5 - (x^2 + 1) = 0$

$-x^2 + 4 = 0$ note: this is the difference of the curves

$x^2 - 4 = (x + 2)(x - 2) = 0$

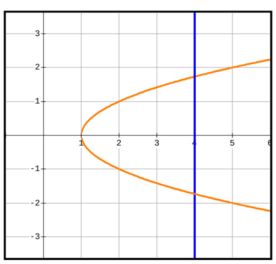
$x = \{-2, 2\}$

Step 3: Integrate the difference of the curves:

Use a definite integral with limits equal to the points of intersection found in Step 2:

$$\int_{-2}^{2} (-x^2 + 4)\, dx \quad \text{or} \quad \int_{-2}^{2} (4 - x^2)\, dx$$

5. $x = y^2 + 1, \quad x = 4$



Step 1: Put the two equations in $x = f(y)$ form:

These two equations are already in this form, so we are good to go.

Step 2: Find the y-values of the points of intersection:

Subtract the function on the left from the function on the right:

$4 - (y^2 + 1) = 0$

$-y^2 + 3 = 0$ note: this is the difference of the curves

$y^2 - 3 = (y - \sqrt{3})(y + \sqrt{3}) = 0$

$y = \{-\sqrt{3}, \sqrt{3}\}$

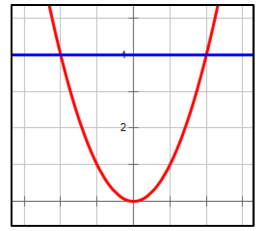
Step 3: Integrate the difference of the curves:

Use a definite integral with limits equal to the points of intersection found in Step 2:

$$\int_{-\sqrt{3}}^{\sqrt{3}} (-y^2 + 3)\, dy \quad \text{or} \quad \int_{-\sqrt{3}}^{\sqrt{3}} (3 - y^2)\, dy$$

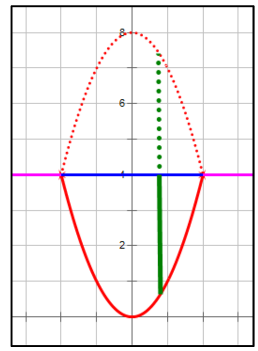
7.2 VOLUME by DISC (circular cross section) or WASHER. Set up the integral (do not evaluate.)

6. $y = x^2$, $y = 4$ a) revolve about $y = 4$ b) revolve about $y = 5$
 c) revolve about $x = 2$ d) revolve about $x = -3$



The points of intersection are obviously $(-2, 4)$ and $(2, 4)$. The region is bounded on the bottom by the x-axis, and on the top by the line $y = 4$.

a) **Revolve about $y = 4$.**



We can use the **disk method** because there is no gap between the region we are revolving and its reflection across the axis of revolution, $y = 4$.

Revolving about a horizontal line means our **disks are vertical**, as shown, and we should **integrate with respect to x**.

The height of the disk is the distance between the curves, $f(x) = 4 - x^2$.

We move the disk from left to right of the region, i.e., **from $x = -2$ to $x = 2$**.

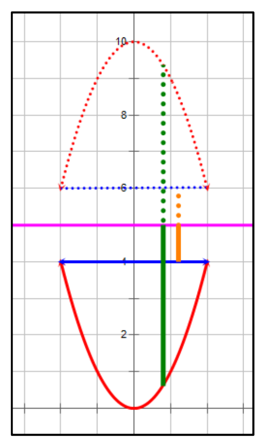
The volume is determined from the formula: $V = \pi \int_a^b (f(x))^2\, dx$. Therefore,

$$V = \pi \int_{-2}^{2} (4 - x^2)^2\, dx$$

Note: For "volume by disk" problems, the curves given to us are expressed in solid lines, while their reflections across the axes of revolution are shown as dotted lines. The disks used to calculate the volumes in the illustrations are either **dark green** or **orange** bars.

b) **Revolve about $y = 5$.**

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $y = 5$.



The washer method is simply a double use of the disk method. We create a large disk and a small disk, and then subtract the two to obtain the "washer."

Revolving about a horizontal line means our **disks are vertical**, as shown, and we should **integrate with respect to x**.

The height of the **larger disk** is the distance between the axis of revolution, $y = 5$ and the curve $y = x^2$, so the height is $f(x) = 5 - x^2$.

The height of the **smaller disk** is the distance between the axis of revolution, $y = 5$ and the curve $y = 4$, so the height is $g(x) = 5 - 4 = 1$.

We move the disks from left to right of the region, i.e., **from $x = -2$ to $x = 2$**.

The volume is determined from the formula: $V = \pi \int_a^b \left[(f(x))^2 - \left(g(x)\right)^2\right] dx$. Therefore,

$$V = \pi \int_{-2}^{2} [(5 - x^2)^2 - 1]\, dx$$

c) **Revolve about $x = 2$.**

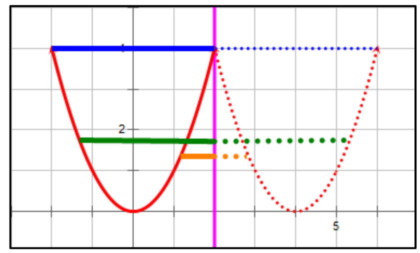
Convert the functions to the form $x = f(y)$:

$y = x^2$ $\qquad$ $y = 4$

$x = \pm\sqrt{y}$ $\qquad$ $y = 4$ (no conversion possible)

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $x = 2$.

Revolving about a vertical line means our **disks are horizontal**, as shown, and we should **integrate with respect to y**.



The width of the **larger disk** is the distance between the axis of revolution, $x = 2$ and the left side of the curve $x = -\sqrt{y}$, so the width is $f(y) = 2 - (-\sqrt{y}) = 2 + \sqrt{y}$.

The width of the **smaller disk** is the distance between the axis of revolution, $x = 2$ and the right side of the curve $x = +\sqrt{y}$, so the width is $g(y) = 2 - \sqrt{y}$.

We move the disks from bottom to top of the region, i.e., **from $y = 0$ to $y = 4$**.

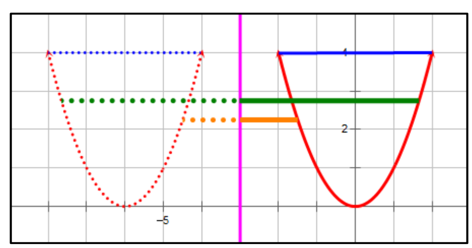
The volume is determined from the formula: $V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2\right] dy$. Therefore,

$$V = \pi \int_0^4 \left[(2+\sqrt{y})^2 - (2-\sqrt{y})^2\right] dy \;=\; 8\pi \int_0^4 \sqrt{y}\, dy$$

d) **Revolve about $x = -3$.**

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $x = -3$.

Revolving about a vertical line means our **disks are horizontal**, as shown, and we should **integrate with respect to y**.



The width of the **larger disk** is the distance between the axis of revolution, $x = -3$ and the right side of the curve $x = +\sqrt{y}$, so the width is $f(y) = 3 + \sqrt{y}$.

The width of the **smaller disk** is the distance between the axis of revolution, $x = -3$ and the left side of the curve $x = -\sqrt{y}$, so the width is $g(y) = 3 - \sqrt{y}$.

We move the disks from bottom to top of the region, i.e., **from $y = 0$ to $y = 4$**.

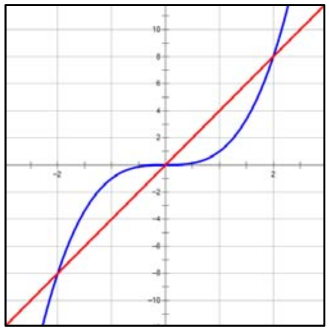
The volume is determined from the formula: $V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2\right] dy$. Therefore,

$$V = \pi \int_0^4 \left[(3+\sqrt{y})^2 - (3-\sqrt{y})^2\right] dy \;=\; 12\pi \int_0^4 \sqrt{y}\, dy$$

7. $y = 4x$, $y = x^3$

a) revolve about $y = 8$

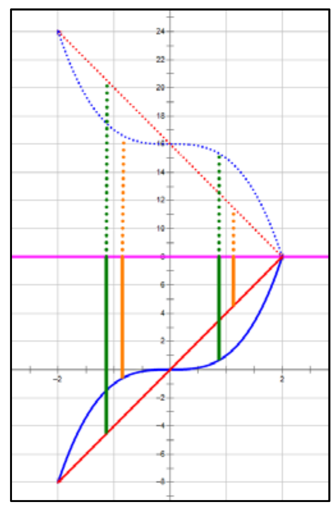
b) revolve about $x = 4$



The points of intersection are obviously $(-2,-8)$, $(0,0)$ and $(2,8)$. The region is bounded on the left and right by the lines $x = \pm 2$, and on the bottom and top by the lines $y = \pm 8$.

a) Revolve about $y = 8$.

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $y = 8$. Additionally, because the curves flip-flop at $x = 0$, we must apply the washer method once for $x \in [-2, 0]$ and once for $x \in [0, 2]$.



Revolving about a horizontal line means our **disks are vertical**, as shown, and we should **integrate with respect to x**.

The height of the **larger disk** is the distance between the axis of revolution, $y = 8$ and the curve farthest away from that line.

- For $x \in [-2, 0]$, the height is $f(x) = 8 - 4x$.
- For $x \in [0, 2]$, the height is $f(x) = 8 - x^3$.

The height of the **smaller disk** is the distance between the axis of revolution, $y = 8$ and the curve closest to that line.

- For $x \in [-2, 0]$, the height is $f(x) = 8 - x^3$.
- For $x \in [0, 2]$, the height is $f(x) = 8 - 4x$.

We move the disks from left to right of the region, i.e., **from $x = -2$ to $x = 2$**.

The volume is determined from the formula: $V = \pi \int_a^b \left[(f(x))^2 - \left(g(x)\right)^2 \right] dx$. We need separate integrals for $x \in [-2, 0]$ and for $x \in [0, 2]$. Therefore,

$$V = \pi \int_{-2}^{0} [(8 - 4x)^2 - (8 - x^3)^2] \, dx \; + \; \pi \int_{0}^{2} [(8 - x^3)^2 - (8 - 4x)^2] \, dx$$

b) Revolve about $x = 4$.

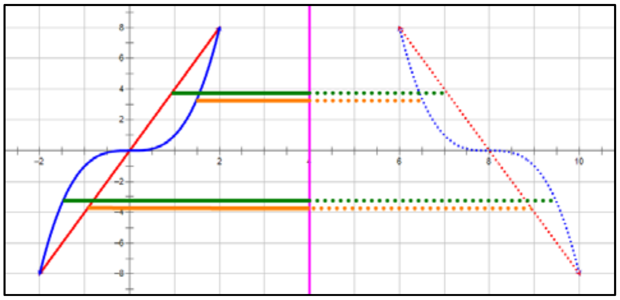
Convert the functions to the form $x = f(y)$:

$y = 4x$ $y = x^3$

$x = \frac{1}{4}y$ $x = \sqrt[3]{y}$

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $x = 4$. Additionally, because the curves flip-flop at $x = 0$, we must apply the washer method once for $y \in [-8, 0]$ and once for $y \in [0, 8]$.

Revolving about a vertical line means our **disks are horizontal**, as shown, and we should **integrate with respect to y**.



The width of the **larger disk** is the distance between the axis of revolution, $x = 4$ and the curve farthest away from that line.

- For $y \in [-8, 0]$, the width is $f(y) = 4 - \sqrt[3]{y}$.
- For $y \in [0, 8]$, the width is $f(y) = 4 - \frac{1}{4}y$.

The width of the **smaller disk** is the distance between the axis of revolution, $x = 4$ and the curve closest to that line.

- For $y \in [-8, 0]$, the height is $f(y) = 4 - \frac{1}{4}y$.
- For $y \in [0, 8]$, the height is $f(y) = 4 - \sqrt[3]{y}$.

We move the disks from bottom to top of the region, i.e., **from $y = -8$ to $y = 8$**.

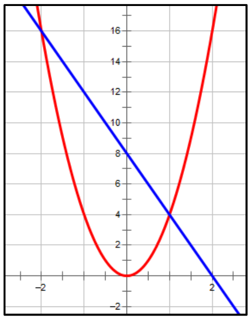
The volume is determined from the formula: $V = \pi \int_c^d \left[(f(y))^2 - \left(g(y)\right)^2 \right] dy$. We need separate integrals for $y \in [-8, 0]$ and once for $y \in [0, 8]$. Therefore,

$$V = \pi \int_{-8}^{0} \left[(4 - \sqrt[3]{y})^2 - \left(4 - \frac{1}{4}y\right)^2 \right] dy \; + \; \pi \int_{0}^{8} \left[\left(4 - \frac{1}{4}y\right)^2 - (4 - \sqrt[3]{y})^2 \right] dy$$

8. $y = 4x^2$, $4x + y - 8 = 0$

   a) revolve about the x-axis

   b) revolve about the line $y = 16$



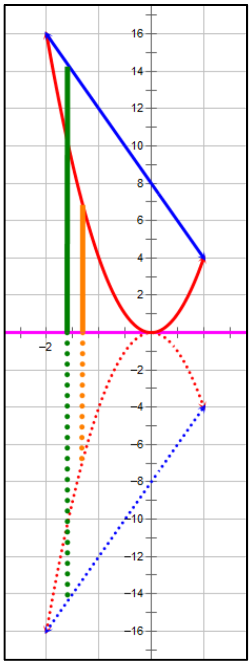
Clean up the equations, as needed:

$y = 4x^2$ $\qquad$ $4x + y - 8 = 0$

$y = 4x^2$ $\qquad$ $y = 8 - 4x$

The points of intersection are obviously $(-2, 16)$ and $(1, 4)$. The region is bounded on the bottom by the x-axis, and on the top by the line $y = 16$.

a) Revolve about the x-axis, i.e., $y = 0$.



We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $y = 0$.

Revolving about a horizontal line means our **disks are vertical**, as shown, and we should **integrate with respect to x**.

The height of the **larger disk** is the distance between the axis of revolution, $y = 0$ and the curve $y = 8 - 4x$, so the height is $f(x) = 8 - 4x$.

The height of the **smaller disk** is the distance between the axis of revolution, $y = 0$ and the curve $y = 4x^2$, so the height is $g(x) = 4x^2$.

We move the disks from left to right of the region, i.e., **from $x = -2$ to $x = 1$**.

The volume is determined from the formula:

$V = \pi \int_a^b \left[(f(x))^2 - \left(g(x)\right)^2 \right] dx$. Therefore,

$$V = \pi \int_{-2}^{1} [(8 - 4x)^2 - (4x^2)^2] \, dx$$

b) Revolve about $y = 16$.

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $y = 16$.

Revolving about a horizontal line means our **disks are vertical**, as shown, and we should **integrate with respect to x**.

The height of the **larger disk** is the distance between the axis of revolution, $y = 16$ and the curve $y = 4x^2$, so the height is $f(x) = 16 - 4x^2$.

The height of the **smaller disk** is the distance between the axis of revolution, $y = 16$ and the curve $y = 8 - 4x$, so the height is $g(x) = 16 - (8 - 4x) = 8 + 4x$.

We move the disks from left to right of the region, i.e., **from $x = -2$ to $x = 1$**.

The volume is determined from the formula:

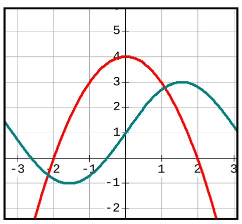
$V = \pi \int_a^b \left[(f(x))^2 - \left(g(x)\right)^2 \right] dx$. Therefore,

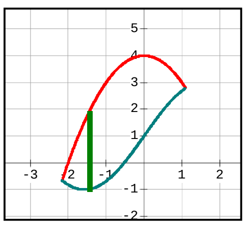
$$V = \pi \int_{-2}^{1} [(16 - 4x^2)^2 - (8 + 4x)^2] \, dx$$

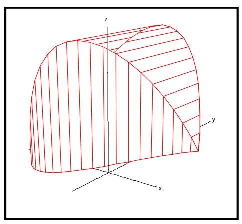
7.2: VOLUME by CROSS-SECTIONS: Set up and evaluate to find the volume. Calculator okay.

9. $y = 4 - x^2$, $y = 1 + 2\sin x$

Find the volume of the solid whose base is bounded by these curves and whose cross-sections perpendicular to the x-axis are squares.







In the formula $V = \int_a^b A(x)\,dx$, $A(x)$ is the formula for the area of the cross section requested in the problem. It has two dimensions, one in the "y-direction" and one coming up out of the page. The third dimension is achieved through integration with respect to x.

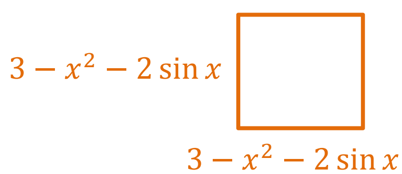
The x-values at the endpoints of the region are $x = \{-2, 1\}$

The difference between the curves is: $(4 - x^2) - (1 + 2\sin x) = 3 - x^2 - 2\sin x$

The formula for the area of a square is $A = s^2$, where s is the length of a side of the square.

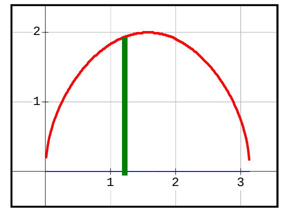
Since the cross sections are squares, we set $s = 3 - x^2 - 2\sin x$, apply the formula for the area of a square and integrate to get the volume:

Cross section:



$$V = \int_{-2}^{1} A(x)\,dx = \int_{-2}^{1} (3 - x^2 - 2\sin x)^2 \, dx \sim 23.940$$

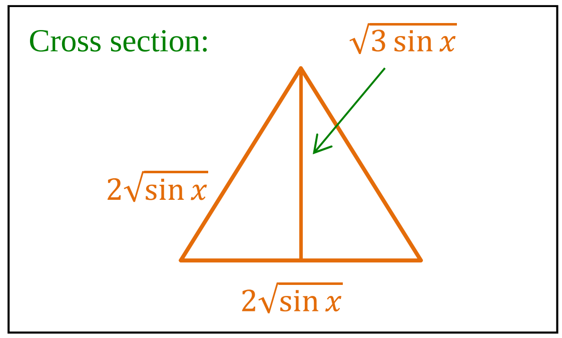
For #10 – 11: $y = 2\sqrt{\sin x}$ $[0, \pi]$ **Find the volume if… Calculator okay.**



10) The cross-sections perpendicular to the x-axis are equilateral triangles with the base of the triangle stretching from the curve to the x-axis.

The x-values at the endpoints of the region are $\boldsymbol{x = \{0, \pi\}}$.

The only curve shown is $\boldsymbol{y = 2\sqrt{\sin x}}$. The other curve is the x-axis, $\boldsymbol{y = 0}$. So, the difference of the curves is $\boldsymbol{y = 2\sqrt{\sin x}}$.



The formula for the area of an equilateral triangle is $\boldsymbol{A = \frac{\sqrt{3}}{4} b^2}$, where $\boldsymbol{b}$ is the length of the base of the triangle.

Since the cross sections are equilateral triangles, we set $\boldsymbol{b = 2\sqrt{\sin x}}$, apply the formula for the area of an equilateral triangle and integrate to get the volume:

$$V = \int_0^{\pi} A(x)\,dx = \int_0^{\pi} \frac{\sqrt{3}}{4}\left(2\sqrt{\sin x}\right)^2 dx$$

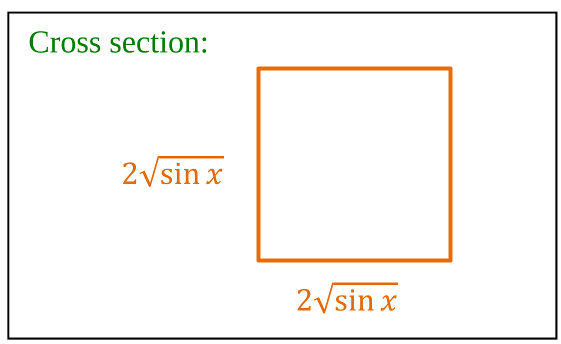
$$V = \sqrt{3}\int_0^{\pi} \sin x \; dx = -\sqrt{3}\;\cos x \,\Big|_0^{\pi}$$

$$\boldsymbol{V = 2\sqrt{3} \sim 3.464}$$

11) The cross-sections perpendicular to the x-axis are squares perpendicular to the x-axis.

The x-values at the endpoints of the region are $\boldsymbol{x = \{0, \pi\}}$.

As in Problem 10 above, the difference of the curves is: $\boldsymbol{y = 2\sqrt{\sin x}}$.



The formula for the area of a square is $\boldsymbol{A = s^2}$, where $\boldsymbol{s}$ is the length of a side of the square.

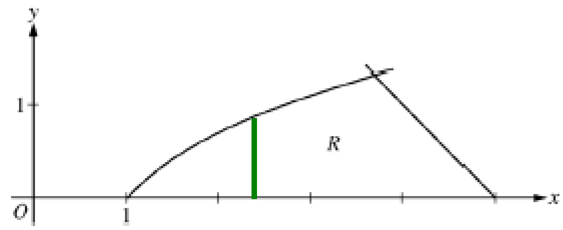
Since the cross sections are squares, we set $\boldsymbol{s = 2\sqrt{\sin x}}$, apply the formula for the area of a square and integrate to get the volume:

$$V = \int_0^{\pi} A(x)\,dx = \int_0^{\pi} \left(2\sqrt{\sin x}\right)^2 \; dx$$

$$V = 4\int_0^{\pi} \sin x \; dx = -4\;\cos x \,\Big|_0^{\pi}$$

$$\boldsymbol{V = 8}$$

12) R is a region bound by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure. Region R is the base of a solid. For the solid, each cross-section perpendicular to the x-axis is a square. Write, but do not evaluate, and integral expression that gives the volume of the solid.



Since the cross sections are described as perpendicular to the x-axis, this solution is developed with respect to x.

First, find the points of intersection of the three curves (including the x-axis).

- Left point of intersection: Set equal $y = \ln x$ and $y = 0$ (the x-axis).
  $\ln x = 0$ has the solution $x = 1$.
- Center point of intersection: Set equal $y = \ln x$ and $y = 5 - x$.
  $\ln x = 5 - x$. This must be solved with a calculator. We get the solution $x = 3.693$.
- Right point of intersection: Set equal $y = 5 - x$ and $y = 0$ (the x-axis).
  $5 - x = 0$ has the solution $x = 5$.

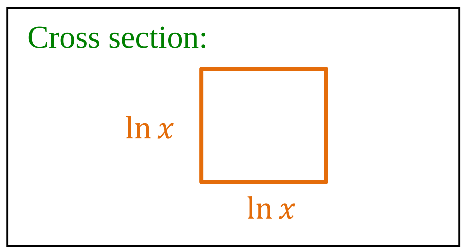
So, we need two integrals, one for $x \in [1, 3.693]$ and one for $x \in [3.693, 5]$.

The formula for the area of a square is $A = s^2$, where s is the length of a side of the square.

Since the cross sections are squares:

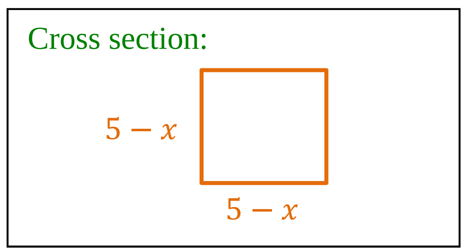
- For $x \in [1, 3.693]$, we set $s = \ln x$. Then, we apply the formula for the area of a square and integrate to get the volume.

  $$V = \int_1^{3.693} A(x)\,dx = \int_1^{3.693} (\ln x)^2\,dx$$



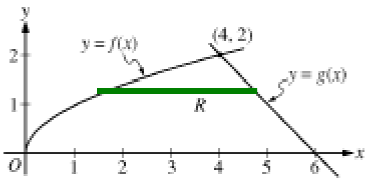
- For $x \in [3.693, 5]$, we set $s = 5 - x$. Then, we apply the formula for the area of a square and integrate to get the volume.

  $$V = \int_{3.693}^{5} A(x)\,dx = \int_{3.693}^{5} (5 - x)^2\,dx$$



The total volume is the sum of the volumes from the two separate integrations.

$$V = \int_1^{3.693} (\ln x)^2\,dx + \int_{3.693}^{5} (5 - x)^2\,dx$$



13) Region R is bound by the x-axis, $f(x) = \sqrt{x}$, and $g(x) = 6 - x$. Region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, and integral expression that gives the volume of the solid.

Since the cross sections are described as perpendicular to the y-axis, this solution is developed with respect to y.

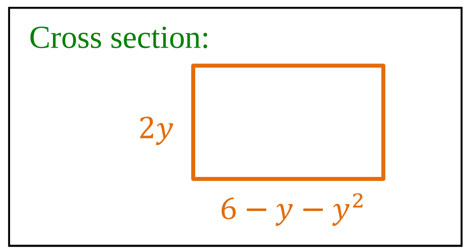
The points of intersection are given as $(0,0)$, $(4,2)$ and $(6,0)$, so we are ahead of the game with respect to those. The y-values of the region range from $y = 0$ to $y = 2$.

We need to revise the function definitions so they are given in terms of $f(y)$.

$y = f(x)$ form: $\quad \boldsymbol{y = \sqrt{x}} \quad \boldsymbol{y = 6 - x}$

$x = f(y)$ form: $\quad \boldsymbol{x = y^2} \quad \boldsymbol{x = 6 - y}$

The formula for the area of a rectangle is $\boldsymbol{A = l \cdot w}$, where $\boldsymbol{l}$ is the length of one side of the rectangle and $\boldsymbol{w}$ is the length of the other side of the rectangle.



In this problem, $\boldsymbol{l = 6 - y - y^2}$, and $\boldsymbol{w = 2y}$ (since the height of the solid is given as $\boldsymbol{2y}$).

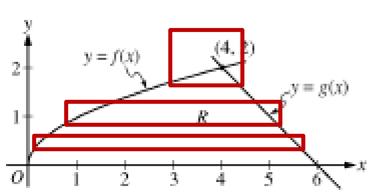
$$\boldsymbol{A = l \cdot w = (6 - y - y^2) \cdot 2y}.$$

The total volume is:

$$V = \int_0^2 A(y)\, dy$$

$$\boldsymbol{V = \int_0^2 (6 - y - y^2) \cdot 2y\ dy = 2\int_0^2 (6y - y^2 - y^3)\ dy}$$

Here's an attempt to show several cross-sections relating to the diagram in this problem. Try to look at it as the rectangles are rising up out of the page. Integrating with respect to the variable y adds up all the rectangles, resulting in the volume of the shape in question.



The width of each rectangle is equal to the distance between the curves. The height in this problem is equal to $2y$.