Worksheet 3. Applying the First Fundamental Theorem of Calculus: Definite Integral as Total Change

Derivation of the First Fundamental Theorem of Calculus

From Second FTC: $\int_a^x f(t)\,dt = F(x) + C$, where F is an antiderivative of f, and

$\int_a^a f(t)\,dt = F(a) + C = 0$, hence, $C = -F(a)$.

Let $x = b$: $\int_a^b f(t)\,dt = F(b) + C$, and, by substitution,

$\int_a^b f(t)\,dt = F(b) - F(a)$ or $F(b) = F(a) + \int_a^b f(t)\,dt$.

Accumulated (or total or net) change is given by the definite integral whose integrand is the rate of change. More specifically, if f is the rate of change of F, then:

$$\int_a^b f(t)dt = \text{Change in } F \text{ from } t = a \text{ to } t = b = F(b) - F(a).$$

Exercises

Write a sentence to answer each of the following questions.

1. If $h(t)$ is the rate of change of the height of a conical pile of sand measured in feet per hour, what does $\int_0^5 h(t)\,dt$ represent? Answer in correct units.

2. If $v(t)$ is the velocity of a particle moving along the x-axis, measured in feet per second, what does $\int_3^{10} v(t)\,dt$ represent? Answer in correct units.

3. If $b(t)$ is the rate of growth of the number of bacteria in a dish, measured in number of bacteria per hour, what does $\int_2^6 b(t)\,dt$ represent? Answer in correct units.

4. If $v(t)$ is the velocity of a particle moving along the x-axis at time t, and the position $x(t)$ is 5 at time $t = 2$, (a) write an integral expression that represents the position of the particle at time $t = 10$, and (b) write an integral expression that gives the total distance traveled by the particle from time $t = 2$ to time $t = 10$.

5. If $p(t)$ is the rate of growth of a rabbit population, measured in rabbits per year, and there were 100 rabbits in the year 2005 ($t = 0$), write an integral expression that represents the rabbit population in 2007.

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Worksheet 4. Example Multiple-Choice Questions

#23. (2003 AB Exam, Section I, Part A, non-calculator section). $\frac{d}{dx}\left(\int_0^{x^2}\sin\left(t^3\right)dt\right) =$

(a) $-\cos(x^6)$ (b) $\sin(x^3)$ (c) $\sin(x^6)$ (d) $2x\sin(x^3)$ (e) $2x\sin(x^6)$

The following questions are from the 2003 AB Exam, Section I, Part B, calculator section.

#82. The rate of change of the altitude of a hot air balloon is given by:

$$r(t) = t^3 - 4t^2 + 6$$

for $0 \le t \le 8$. Which of the following expressions gives the change in altitude of the balloon during the time the altitude is decreasing?

(a) $\int_{1.572}^{3.514} r(t)dt$ (b) $\int_0^8 r(t)dt$ (c) $\int_0^{2.667} r(t)dt$ (d) $\int_{1.572}^{3.514} r'(t)dt$ (e) $\int_0^{2.667} r'(t)dt$

#91. A particle moves along the x-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln(1+2^t)$. If the velocity of the particle is 2 at time $t = 1$, then the velocity of the particle at time $t = 2$ is

(a) 0.462 (b) 1.609 (c) 2.555 (d) 2.886 (e) 3.346

#92. Let g be the function given by:

$$g(x) = \int_0^x \sin\left(t^2\right)dt$$

for $-1 \le x \le 3$. On which of the following intervals is g decreasing?

(a) $-1 \le x \le 0$ (b) $0 \le x \le 1.772$ (c) $1.253 \le x \le 2.171$ (d) $1.772 \le x \le 2.507$ (e) $2.802 \le x \le 3$

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Worksheet 5. Applying the Fundamental Theorem of Calculus: Exercises

Work Problems 1–3 by both methods.

1. $y' = 2 + \frac{1}{x^2}$ and $y(1) = 6$. Find $y(3)$

2. $f'(x) = \cos(2x)$ and $f(0) = 3$. Find $f\left(\frac{\pi}{4}\right)$.

3. Water flows into a tank at a rate of $\frac{dW}{dt} = \frac{1}{75}\left(600 + 20t - t^2\right)$ where $\frac{dW}{dt}$ is measured in gallons per hour and t is measured in hours. If there are 150 gallons of water in the tank at time $t = 0$, how many gallons of water are in the tank when $t = 24$?

Work Problems 4–10 using the Fundamental Theorem of Calculus and your calculator.

4. $f'(x) = \cos\left(x^3\right)$and $f(0) = 2$. Find $f(1)$

5. $f'(x) = e^{-x^2}$ and $f(5) = 1$. Find $f(2)$

6. A particle moving along the x-axis has position $x(t)$ at time t with the velocity of the particle given by $v(t) = 5\sin\left(t^2\right)$ At time $t = 6$, the particle's position is (4, 0). Find the position of the particle when $t = 7$.

7. Let $F(t)$ represent a bacteria population which is 4 million at time $t = 0$. After t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Find the total increase in the bacteria population during the first three hours, and find the population at $t = 3$ hours.

8. A particle moves along a line so that at any time $t \geq 0$ its velocity is given by $v(t) = \frac{t}{1 + t^2}$. At time $t = 0$, the position of the particle is $s(0) = 5$. Determine the position of the particle at $t = 3$.

9. Let f be the function whose graph passes through the point (3, 6) and whose derivative is given by:

$$f'(x) = \frac{1+e^x}{x^2}. \text{ Find } f(3.1).$$

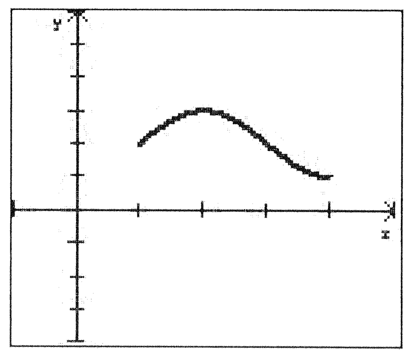
10. (Multiple Choice) If f is the antiderivative of $\frac{x^2}{1+x^5}$ such that $f(1) = 5$, then $f(4) =$

(a) 4.988 (b) 5 (c) 5.016 (d) 5.376 (e) 5.629

In Problems 11–13, use the Fundamental Theorem of Calculus and the given graph. Each tick mark on the axes below represents one unit.

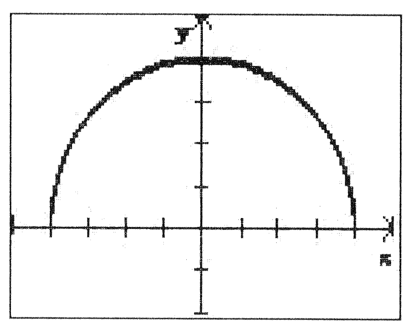
11. The graph of f' is shown at right.

$\int_1^4 f'(x)dx = 6.2$ and $f(1) = 3$. Find $f(4)$.



12. The graph of f' is the semicircle shown at right.

Find $f(-4)$ given that $f(4) = 7$.

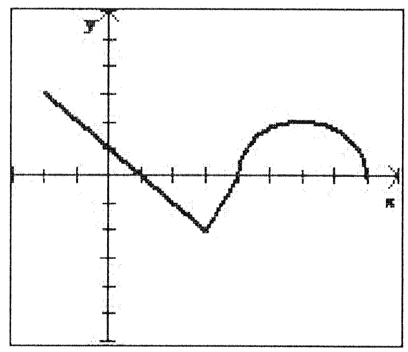


13. The graph of f', consisting of two line segments and a semicircle, is shown at right. Given that $f(-2) = 5$, find:

(a) $f(1)$

(b) $f(4)$

(c) $f(8)$



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