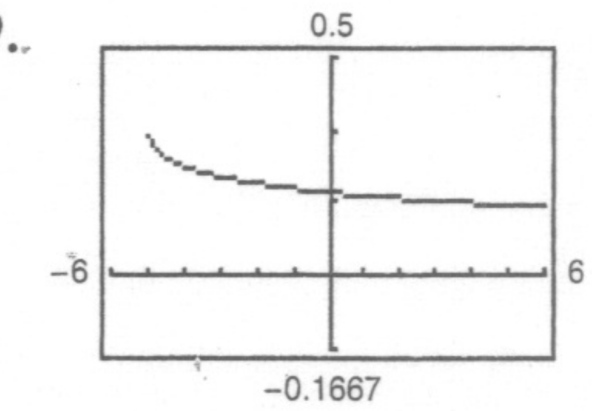
49.

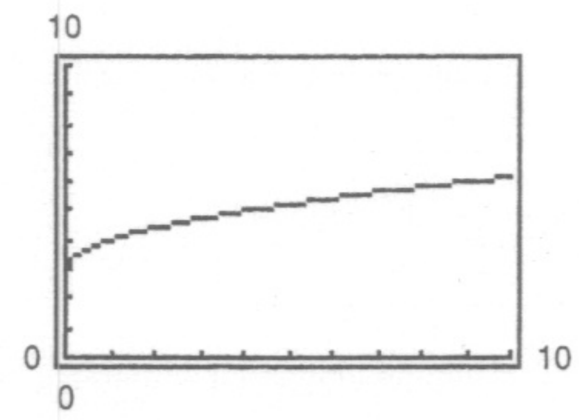


$\lim_{x\to 4} f(x) = \frac{1}{6}$

Domain: $[-5, 4) \cup (4, \infty)$

The graph has a hole at $x = 4$.

51.



$\lim_{x\to 9} f(x) = 6$

Domain: $[0, 9) \cup (9, \infty)$

The graph has a hole at $x = 9$.

53. Answers will vary. Sample answer: As x approaches 8 from either side, $f(x)$ becomes arbitrarily close to 25.

55. No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

57. (a) $r = \dfrac{3}{\pi} \approx 0.9549$ cm

(b) $\dfrac{5.5}{2\pi} \le r \le \dfrac{6.5}{2\pi}$, or approximately $0.8754 < r < 1.0345$

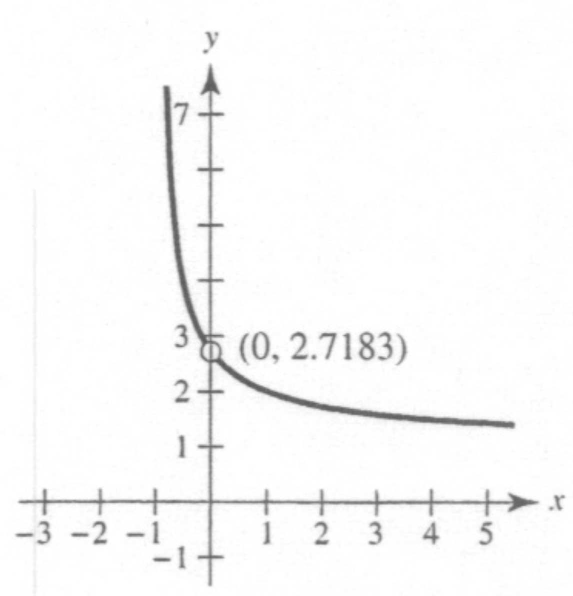
(c) $\lim_{r\to 3/\pi} 2\pi r = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

59.

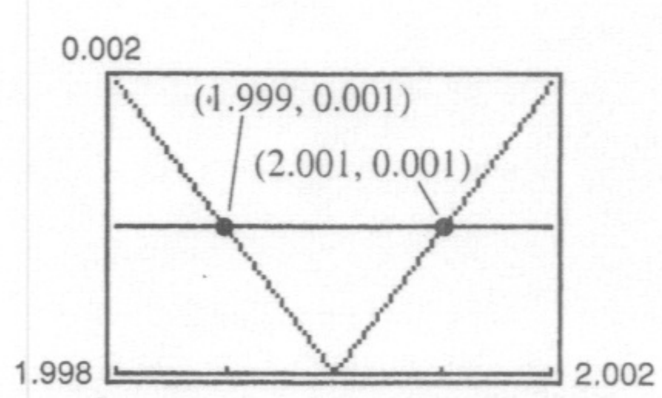
x	−0.001	−0.0001	−0.00001
$f(x)$	2.7196	2.7184	2.7183

x	0.00001	0.0001	0.001
$f(x)$	2.7183	2.7181	2.7169

$\lim_{x\to 0} f(x) \approx 2.7183$



61.



$\delta \approx 0.001$

$(1.999, 2.001)$

63. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

65. False. See Exercise 11.

67. (a) Yes. As x approaches 0.25 from either side, $\sqrt{x}$ becomes arbitrarily close to 0.5.

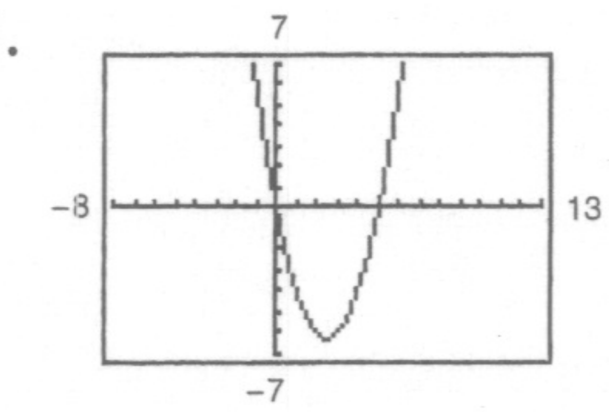
(b) No. $\lim_{x\to 0} \sqrt{x}$ does not exist because for $x < 0$, $\sqrt{x}$ does not exist.

69–71. Proofs **73.** Answers will vary.

75. Putnam Problem B1, 1986

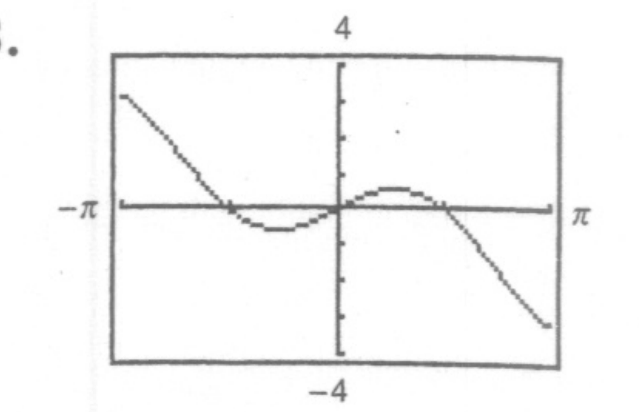
Section 1.3 (page 67)

1.



(a) 0 (b) 6

3.



(a) 0 (b) ≈ 0.52 or $\pi/6$

5. 16 **7.** −1 **9.** 0 **11.** 7 **13.** 1/2 **15.** −2/5

17. 35/3 **19.** 2 **21.** 1 **23.** (a) 4 (b) 64 (c) 64

25. (a) 3 (b) 2 (c) 2 **27.** 1 **29.** −1/2 **31.** 1

33. 1/2 **35.** −1 **37.** (a) 15 (b) 5 (c) 6 (d) 2/3

39. (a) 64 (b) 2 (c) 12 (d) 8

41. (a) 1 (b) 3

$g(x) = \dfrac{-2x^2 + x}{x}$ and $f(x) = -2x + 1$ agree except at $x = 0$.

43. (a) 2 (b) 0

$g(x) = \dfrac{x^3 - x}{x - 1}$ and $f(x) = x^2 + x$ agree except at $x = 1$.

45. −2

$f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

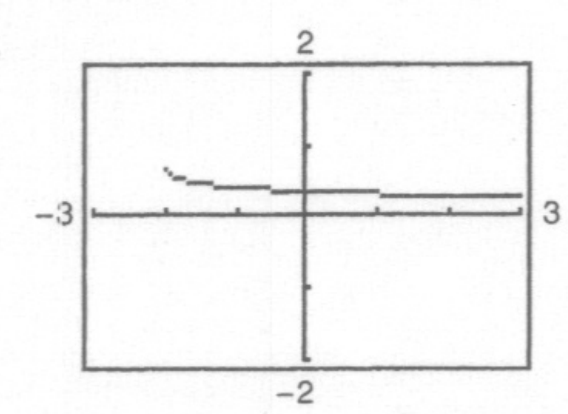
47. 12

$f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

49. 1/10 **51.** 5/6 **53.** $\sqrt{5}/10$ **55.** 1/6 **57.** −1/9

59. 2 **61.** $2x - 2$

63.



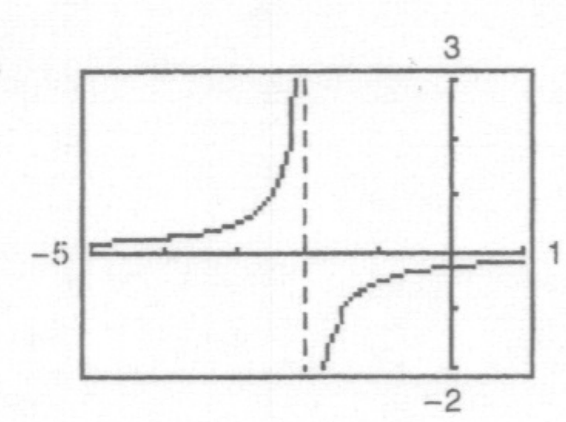
The graph has a hole at $x = 0$.

Answers will vary. Example:

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$\lim_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$ $\left(\text{Actual limit is } \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}.\right)$

65.



The graph has a hole at $x = 0$.

Answers will vary. Example:

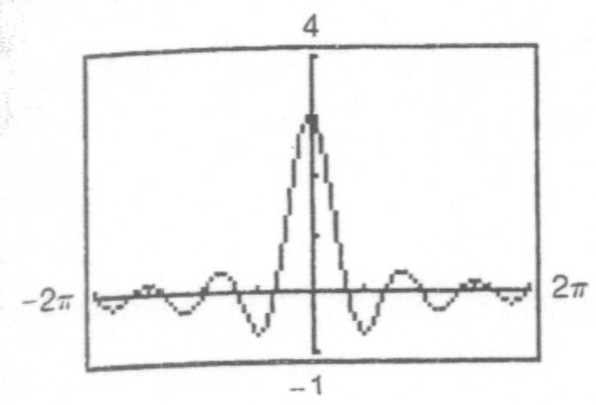
x	−0.1	−0.01	−0.001
$f(x)$	−0.263	−0.251	−0.250

x	0.001	0.01	0.1
$f(x)$	−0.250	−0.249	−0.238

$\lim_{x\to 0} \dfrac{[1/(2+x)] - (1/2)}{x} \approx -0.250$ $\left(\text{Actual limit is } -\dfrac{1}{4}.\right)$

67. 1/5 **69.** 0 **71.** 0 **73.** 0 **75.** 1 **77.** 3/2

79.



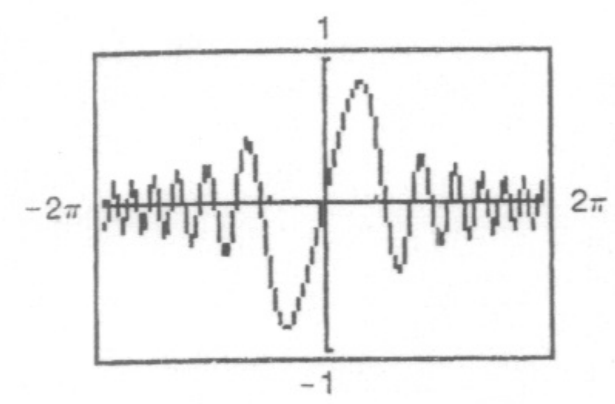
The graph has a hole at $t = 0$.

Answers will vary. Example:

t	−0.1	−0.01	0	0.01	0.1
$f(t)$	2.96	2.9996	?	2.9996	2.96

$$\lim_{t\to 0} \frac{\sin 3t}{t} = 3$$

81.



The graph has a hole at $x = 0$.

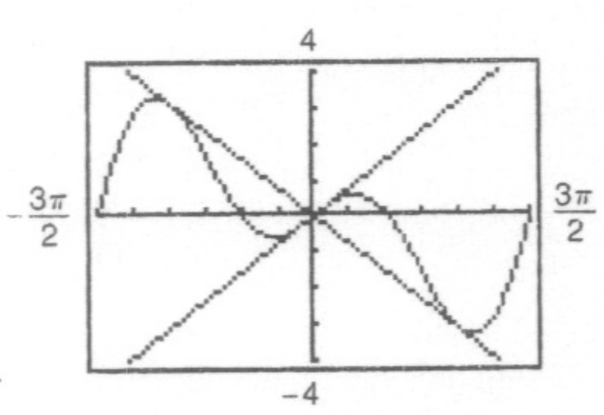
Answers will vary. Example:

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	−0.1	−0.01	−0.001	?	0.001	0.01	0.1

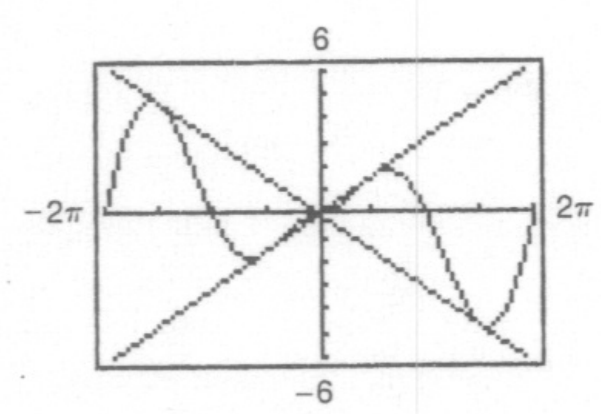
$$\lim_{x\to 0} \frac{\sin x^2}{x} = 0$$

83. 2 **85.** $-4/x^2$ **87.** 4

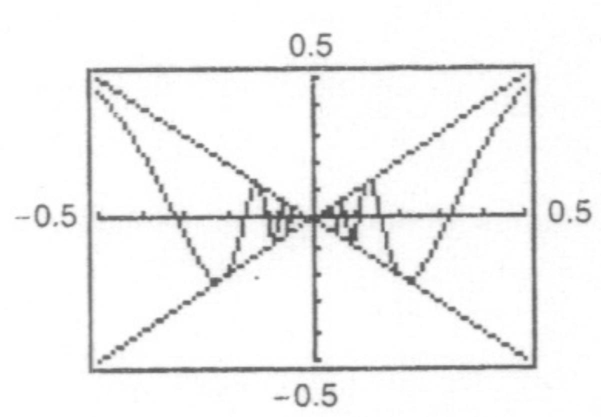
89. 0



91. 0



93. 0

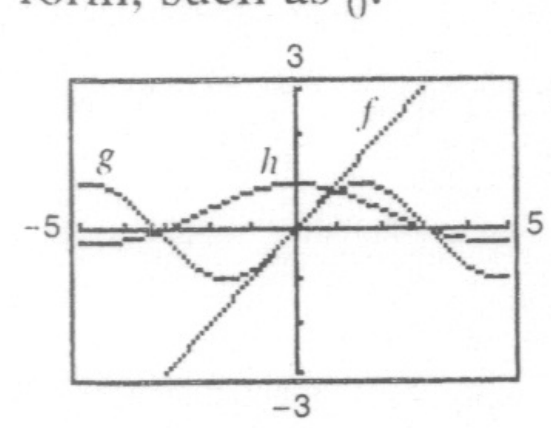


The graph has a hole at $x = 0$.

95. f and g agree at all but one point if c is a real number such that $f(x) = g(x)$ for all $x \neq c$.

97. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional form, such as $\frac{0}{0}$.

99.



The magnitudes of $f(x)$ and $g(x)$ are approximately equal when x is close to 0. Therefore, their ratio is approximately 1.

101. 160 ft/sec **103.** −29.4 m/sec

105. Let $f(x) = 1/x$ and $g(x) = -1/x$.

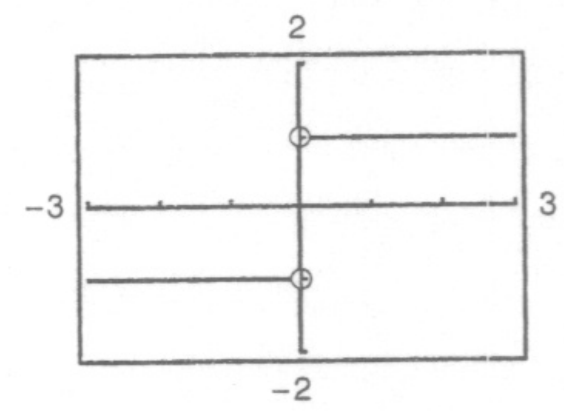
$\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. However,

$$\lim_{x\to 0} [f(x) + g(x)] = \lim_{x\to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to 0} 0 = 0$$

and therefore does exist.

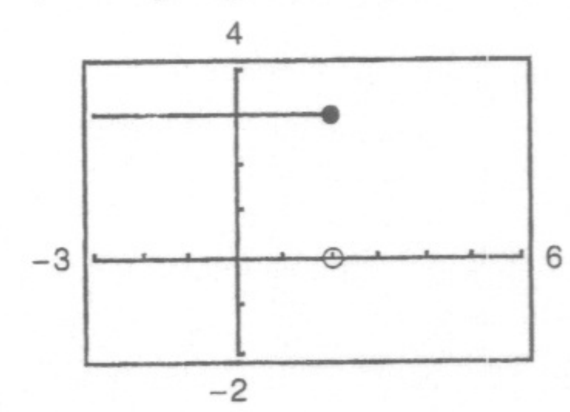
107–111. Proofs

113. False. The limit does not exist because the function approaches 1 from the right side of 0 and approaches −1 from the left side of 0. (See graph below.)



115. True.

117. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2. (See graph below.)



119. Let $f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$

$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

121. $\lim_{x\to 0} f(x)$ does not exist because $f(x)$ oscillates between two fixed values as x approaches 0.

$\lim_{x\to 0} g(x) = 0$ because, as x gets increasingly closer to 0, the values of $g(x)$ become increasingly closer to 0.

123. (a) 1/2

(b) Because $\dfrac{1 - \cos x}{x^2} \approx \dfrac{1}{2}$, it follows that $1 - \cos x \approx \dfrac{1}{2}x^2$

$\cos x \approx 1 - \dfrac{1}{2}x^2$ when $x \approx 0$.

(c) 0.995 (d) Calculator: $\cos(0.1) \approx 0.9950$

Section 1.4 (page 78)

1. (a) 1 (b) 1 (c) 1; $f(x)$ is continuous on $(-\infty, \infty)$.

3. (a) 0 (b) 0 (c) 0; Discontinuity at $x = 3$

5. (a) 2 (b) −2 (c) Limit does not exist. Discontinuity at $x = 4$

7. $\frac{1}{10}$

9. Limit does not exist. The function decreases without bound as x approaches −3 from the left.

11. −1 **13.** $-1/x^2$ **15.** 5/2 **17.** 2

19. Limit does not exist. The function decreases without bound as x approaches π from the left and increases without bound as x approaches π from the right.

21. 4

23. Limit does not exist. The function approaches 5 from the left side of 3 but approaches 6 from the right side of 3.

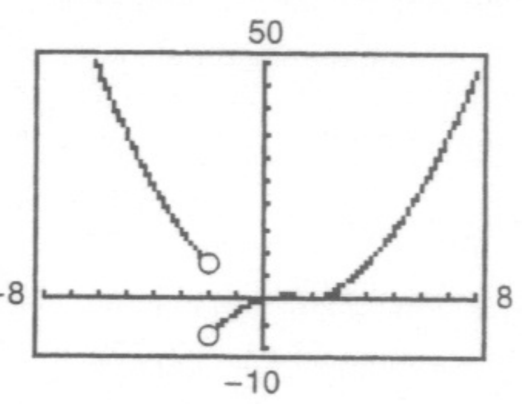
25. Discontinuous at $x = -2$ and $x = 2$

27. Discontinuous at every integer

29. Continuous on $[-5, 5]$ **31.** Continuous on $[-1, 4]$

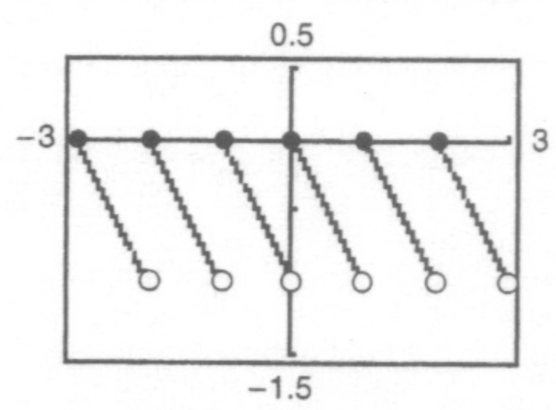
33. Continuous for all real x **35.** Continuous for all real x

37. Nonremovable discontinuity at $x = 1$
Removable discontinuity at $x = 0$
39. Continuous for all real x
41. Removable discontinuity at $x = -2$
Nonremovable discontinuity at $x = 5$
43. Nonremovable discontinuity at $x = -2$
45. Continuous for all real x
47. Nonremovable discontinuity at $x = 2$
49. Continuous for all real x
51. Nonremovable discontinuities at integer multiples of $\pi/2$
53. Nonremovable discontinuity at each integer
55.



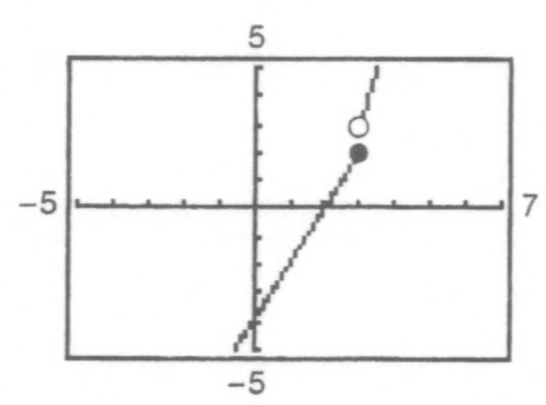
$\lim_{x\to 0^+} f(x) = 0$
$\lim_{x\to 0^-} f(x) = 0$
Discontinuity at $x = -2$

57. $a = 2$ **59.** $a = -1,\ b = 1$ **61.** Continuous for all real x
63. Nonremovable discontinuities at $x = 1$ and $x = -1$
65.



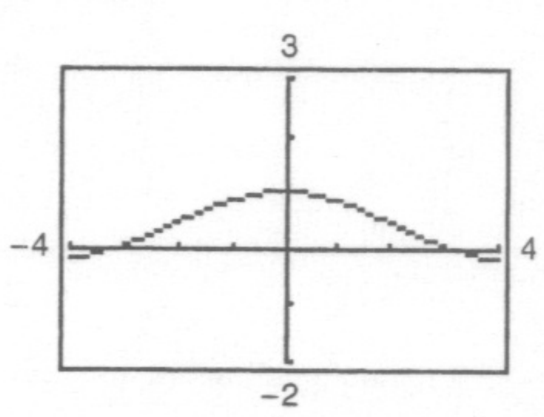
Nonremovable discontinuity at each integer

67.



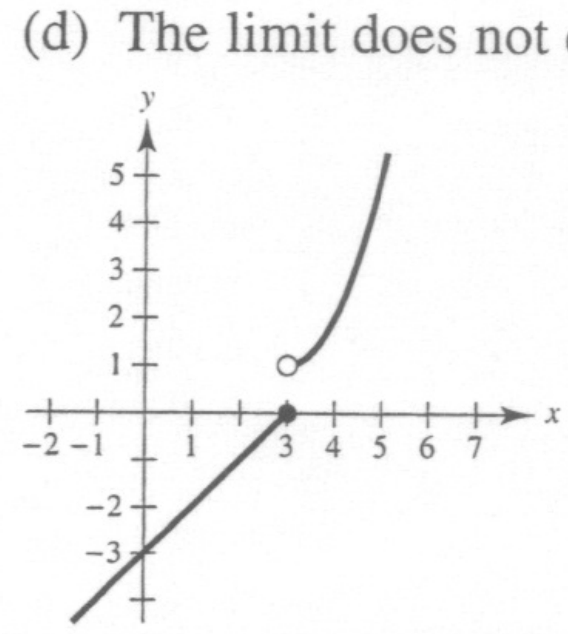
Discontinuous at $x = 3$

69. Continuous on $(-\infty, \infty)$
71. Continuous on the open intervals . . . $(-6, -2)$, $(-2, 2)$, $(2, 6)$, ...
73.



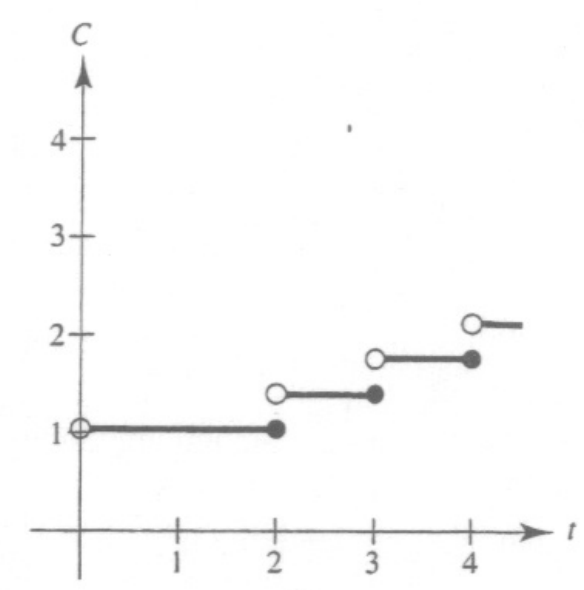
The graph has a hole at $x = 0$. The graph appears continuous but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 0$.

75. Because $f(x)$ is continuous on the interval $[1, 2]$ and $f(1) = 2.0625$ and $f(2) = -4$, by the Intermediate Value Theorem there exists a real number c in $[1, 2]$ such that $f(c) = 0$.
77. Because $f(x)$ is continuous on the interval $[0, \pi]$ and $f(0) = -3$ and $f(\pi) \approx 8.87$, by the Intermediate Value Theorem there exists a real number c in $[0, \pi]$ such that $f(c) = 0$.
79. 0.68, 0.6823 **81.** 0.56, 0.5636
83. $f(3) = 11$ **85.** $f(2) = 4$
87. (a) The limit does not exist at $x = c$.
(b) The function is not defined at $x = c$.
(c) The limit exists, but it is not equal to the value of the function at $x = c$.
(d) The limit does not exist at $x = c$.
89.



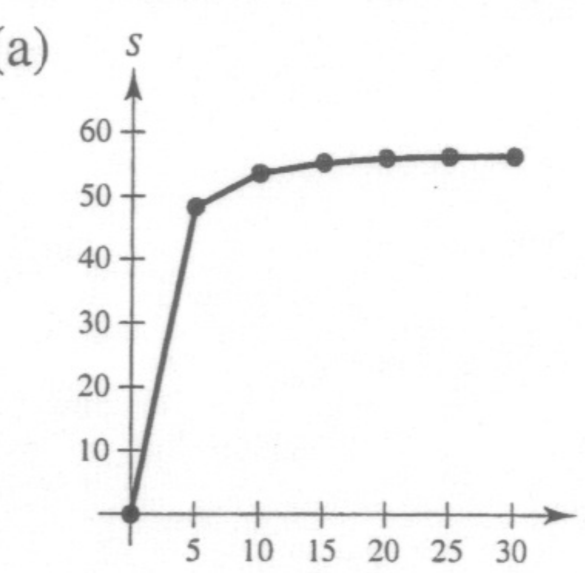
Not continuous because $\lim_{x\to 3} f(x)$ does not exist.

91. True
93. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.
95. $\lim_{t\to 4^-} f(t) \approx 28$; $\lim_{t\to 4^+} f(t) \approx 56$
At the end of day 3, the amount of chlorine in the pool is about 28 oz. At the beginning of day 4, the amount of chlorine in the pool is about 56 oz.
97. $C = \begin{cases} 1.04, & 0 < t \le 2 \\ 1.04 + 0.36[\![t - 1]\!], & t > 2,\ t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2,\ t \text{ is an integer} \end{cases}$



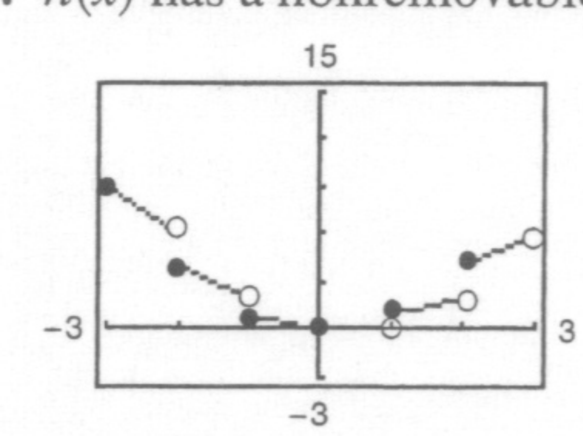
Nonremovable discontinuity at each integer greater than or equal to 2.

99–101. Proofs **103.** Answers will vary.
105. (a)



(b) There appears to be a limiting speed, and a possible cause is air resistance.

107. $c = (-1 \pm \sqrt{5})/2$
109. Domain: $[-c^2, 0) \cup (0, \infty)$; Let $f(0) = 1/(2c)$
111. $h(x)$ has a nonremovable discontinuity at every integer except 0.



113. Putnam Problem B2, 1988

Section 1.5 (page 88)

1. $\lim_{x\to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$ $\lim_{x\to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$
3. $\lim_{x\to -2^+} \tan((\pi x)/4) = -\infty$ $\lim_{x\to -2^-} \tan((\pi x)/4) = \infty$
5.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	0.31	1.64	16.6	167

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-167	-16.7	-1.69	-0.36

$\lim_{x\to -3^+} f(x) = -\infty$ $\lim_{x\to -3^-} f(x) = \infty$

7.

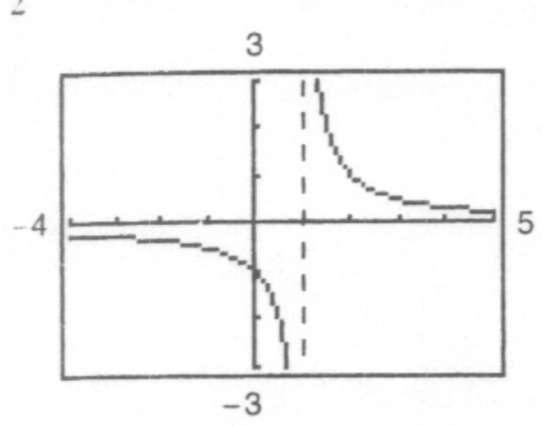
x	-3.5	-3.1	-3.01	-3.001
$f(x)$	3.8	16	151	1501

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1499	-149	-14	-2.3

$\lim_{x\to -3^+} f(x) = -\infty \quad \lim_{x\to -3^-} f(x) = \infty$

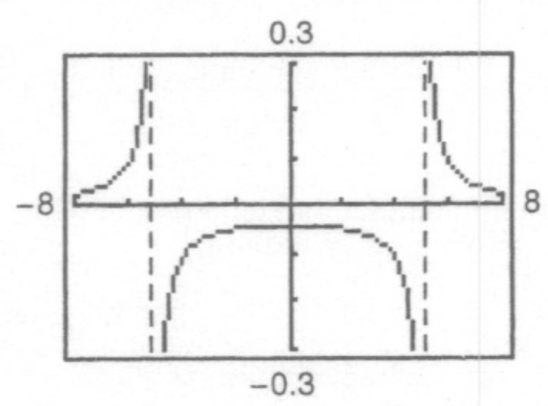
9. $x = 0$ **11.** $x = 2,\ x = -1$ **13.** $x = \pm 2$
15. No vertical asymptote **17.** $x = \pi/4 + (n\pi)/2$, n is an integer.
19. $t = 0$ **21.** $x = -2, x = 1$ **23.** No vertical asymptote
25. No vertical asymptote **27.** $t = n\pi$, n is a nonzero integer.
29. Removable discontinuity at $x = -1$
31. Vertical asymptote at $x = -1$ **33.** $-\infty$ **35.** ∞ **37.** $\frac{4}{5}$
39. $\frac{1}{2}$ **41.** $-\infty$ **43.** ∞ **45.** 0 **47.** Does not exist

49.



$\lim_{x\to 1^+} f(x) = \infty$

51.

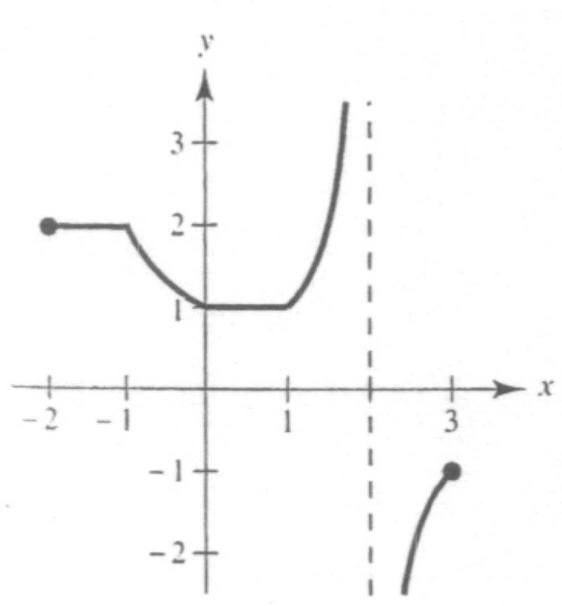


$\lim_{x\to 5^-} f(x) = -\infty$

53. Answers will vary.

55. Answers will vary. Example: $f(x) = \dfrac{x-3}{x^2 - 4x - 12}$

57.



59. (a) $\frac{1}{3}(200\pi)$ ft/sec
(b) 200π ft/sec
(c) $\lim_{\theta\to(\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

61. ∞

63. (a) Domain: $x > 25$
(b)

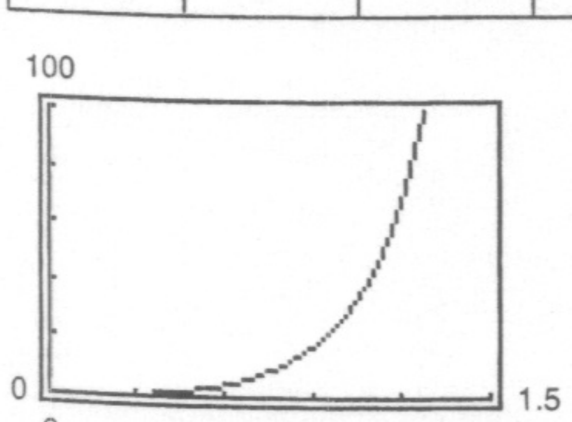
x	30	40	50	60
y	150	66.667	50	42.857

As x gets close to 25 mph, y becomes larger and larger.

(c) $\lim_{x\to 25^+} \dfrac{25x}{x-25} = \infty$

65. (a) $A = 50 \tan\theta - 50\theta$; Domain: $(0, \pi/2)$
(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\lim_{\theta\to\pi/2^-} A = \infty$

67. False; let $f(x) = (x^2 - 1)/(x - 1)$ **69.** True

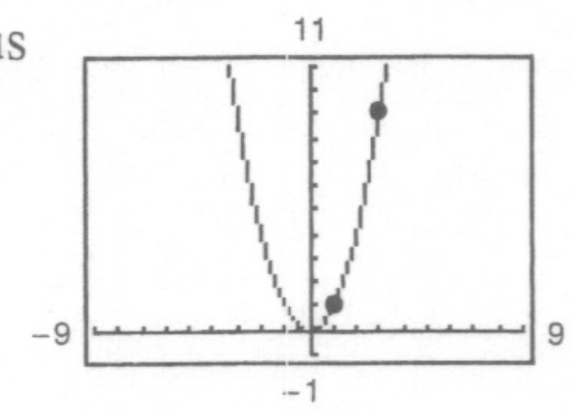
71. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$. $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x\to 0} \dfrac{1}{x^4} = \infty$, but $\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim_{x\to 0}\left(\dfrac{x^2-1}{x^4}\right) = -\infty \neq 0$.

73. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

75. Answers will vary.

Review Exercises for Chapter 1 (page 91)

1. Calculus



Estimate: 8.261

3.

x	-0.1	-0.01	-0.001
$f(x)$	-1.0526	-1.0050	-1.0005

x	0.001	0.01	0.1
$f(x)$	-0.9995	-0.9950	-0.9524

The estimate of the limit of $f(x)$, as x approaches zero, is -1.00.

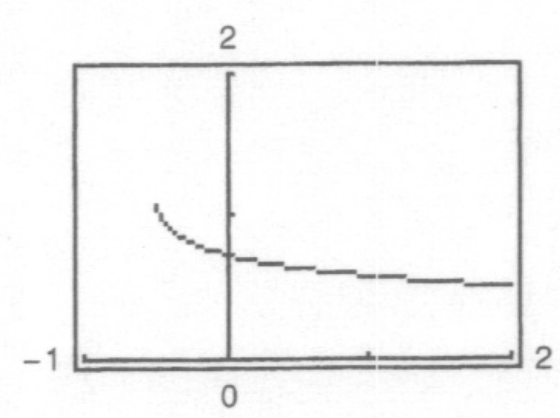
5. (a) -2 (b) -3 **7.** 2; Proof **9.** 1; Proof
11. $\sqrt{6} \approx 2.45$ **13.** $-\frac{1}{4}$ **15.** $\frac{1}{4}$ **17.** -1 **19.** 75
21. 0 **23.** $\sqrt{3}/2$ **25.** $-\frac{1}{2}$

27. (a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$\lim_{x\to 1^+} f(x) \approx 0.5773$

(b)



The graph has a hole at $x = 1$.
$\lim_{x\to 1^+} f(x) \approx 0.5774$

(c) $\sqrt{3}/3$

29. -39.2 m/sec **31.** -1 **33.** 0
35. Limit does not exist. The limit as t approaches 1 from the left is 2 whereas the limit as t approaches 1 from the right is 1.
37. Nonremovable discontinuity at each integer
Continuous on $(k, k + 1)$ for all integers k
39. Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$
41. Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$
43. Nonremovable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$
45. Nonremovable discontinuity at each even integer
Continuous on $(2k, 2k + 2)$ for all integers k
47. $c = -\frac{1}{2}$ **49.** Proof
51. (a) -4 (b) 4 (c) Limit does not exist.
53. $x = 0$ **55.** $x = 10$ **57.** $-\infty$ **59.** $\frac{1}{3}$
61. $-\infty$ **63.** $-\infty$ **65.** $\frac{4}{5}$ **67.** ∞
69. (a) $14,117.65 (b) $80,000.00 (c) $720,000.00 (d) ∞

P.S. Problem Solving (page 93)

1. (a) Perimeter $\triangle PAO = 1 + \sqrt{(x^2-1)^2 + x^2} + \sqrt{x^4 + x^2}$
Perimeter $\triangle PBO = 1 + \sqrt{x^4 + (x-1)^2} + \sqrt{x^4 + x^2}$