



AP^{*} Calculus Review

Limits, Continuity, and the Definition of the Derivative

Teacher Packet

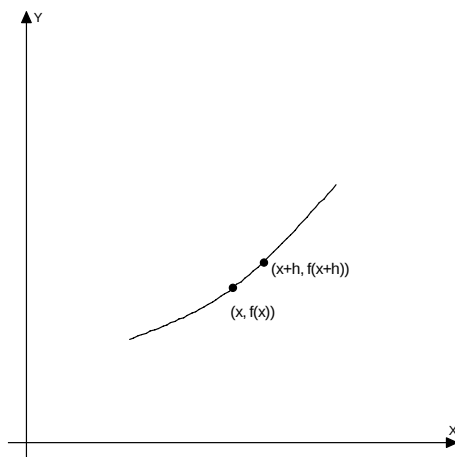
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DEFINITION Derivative of a Function

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$



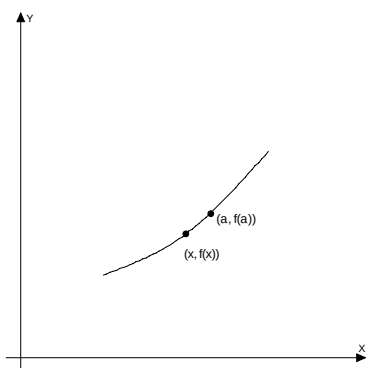
provided the limit exists.

You will want to recognize this formula (a slope) and know that you need to take the derivative of $f(x)$ when you are asked to find $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

DEFINITION (ALTERNATE) Derivative at a Point

The **derivative** of the function f **at the point** $x = a$ is the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$



provided the limit exists.

This is the slope of a segment connecting two points that are very close together.

DEFINITION Continuity

A function f is continuous at a number a if

- 1) $f(a)$ is defined (a is in the domain of f)
- 2) $\lim_{x \rightarrow a} f(x)$ exists
- 3) $\lim_{x \rightarrow a} f(x) = f(a)$

A function is continuous at an x if the function has a value at that x , the function has a limit at that x , and the value and the limit are the same.

Example:

$$\text{Given } f(x) = \begin{cases} x^2 + 3, & x \leq 2 \\ 3x + 2, & x > 2 \end{cases}$$

Is the function continuous at $x = 2$?

$$f(x) = 7$$

$$\lim_{x \rightarrow 2^-} f(x) = 7, \text{ but the } \lim_{x \rightarrow 2^+} f(x) = 8$$

The function does not have a limit as $x \rightarrow 2$, therefore the function is not continuous at $x = 2$.

Limits as x approaches ∞

For rational functions, examine the x with the largest exponent, numerator and denominator. The x with the largest exponent will carry the weight of the function.

If the x with the largest exponent is in the denominator, the denominator is growing faster as $x \rightarrow \infty$. Therefore, the limit is 0.

$$\lim_{x \rightarrow \infty} \frac{3 + x}{x^4 - 3x + 7} = 0$$

If the x with the largest exponent is in the numerator, the numerator is growing faster as $x \rightarrow \infty$. The function behaves like the resulting function when you divide the x with the largest exponent in the numerator by the x with the largest exponent in the denominator.

$$\lim_{x \rightarrow \infty} \frac{3 + x^5}{x^2 - 3x + 7} = \infty$$

This function has end behavior like $x^3 \left(\frac{x^5}{x^2} \right)$. The function does not reach a limit, but to say the limit equals infinity gives a very good picture of the behavior.

If the x with the largest exponent is the same, numerator and denominator, the limit is the coefficients of the two x 's with that largest exponent.

$$\lim_{x \rightarrow \infty} \frac{3 + 4x^5}{7x^5 - 3x + 7} = \frac{4}{7}. \text{ As } x \rightarrow \infty, \text{ those } x^5 \text{ terms are like gymnasiums full of sand.}$$

The few grains of sand in the rest of the function do not greatly affect the behavior of the function as $x \rightarrow \infty$.

LIMITS

$$\lim_{x \rightarrow c} f(x) = L$$

The limit of f of x as x approaches c equals L .

As x gets closer and closer to some number c (but does not equal c), the value of the function gets closer and closer (and may equal) some value L .

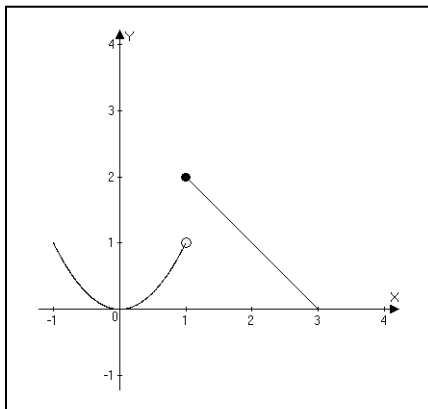
One-sided Limits

$$\lim_{x \rightarrow c^-} f(x) = L$$

The limit of f of x as x approaches c from the left equals L .

$$\lim_{x \rightarrow c^+} f(x) = L$$

The limit of f of x as x approaches c from the right equals L .



Using the graph above, evaluate the following:

$$\lim_{x \rightarrow 1^-} f(x) =$$

$$\lim_{x \rightarrow 1^+} f(x) =$$

$$\lim_{x \rightarrow 1} f(x) =$$

Practice Problems

Limit as x approaches infinity

1. $\lim_{x \rightarrow \infty} \left(\frac{3x - 7}{5x^4 - 8x + 12} \right) =$

2. $\lim_{x \rightarrow \infty} \left(\frac{3x^4 - 2}{5x^4 - 2x + 1} \right) =$

3. $\lim_{x \rightarrow \infty} \left(\frac{x^6 - 2}{10x^4 - 9x + 8} \right) =$

4. $\lim_{x \rightarrow \infty} \left(\frac{7x^4 - 2}{5 - 2x^3 - 14x^4} \right) =$

5. $\lim_{x \rightarrow \infty} \left(\frac{\sin x}{e^x} \right) =$

6. $\lim_{x \rightarrow -\infty} \left(\frac{\sqrt{x^2 - 9}}{2x - 3} \right) =$

7. $\lim_{x \rightarrow \infty} \left(\frac{\sqrt{x^2 - 9}}{2x - 3} \right) =$

Practice Problems

Limit as x approaches a number

8. $\lim_{x \rightarrow 2} (x^3 - x + 1)$

9. $\lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x - 2} \right) =$

10. $\lim_{x \rightarrow 2^-} \left(\frac{3}{x - 2} \right) =$

11. $\lim_{x \rightarrow 2^+} \left(\frac{3}{x - 2} \right) =$

12. $\lim_{x \rightarrow 2} \left(\frac{3}{x - 2} \right) =$

13. $\lim_{x \rightarrow 2^+} \left(\frac{3}{2 - x} \right) =$

14. $\lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\sin x}{x} \right) =$

15. $\lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\tan x}{x} \right) =$

1. What is $\lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$?

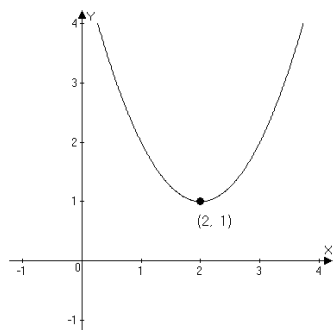
- (A) $\sin x$ (B) $\cos x$ (C) $-\sin x$
(D) $-\cos x$ (E) The limit does not exist

2. $\lim_{\Delta x \rightarrow 0} \frac{\cos\left(\frac{\pi}{3} + \Delta x\right) - \cos\left(\frac{\pi}{3}\right)}{\Delta x} =$

- (A) $-\frac{\sqrt{3}}{2}$ (B) $-\frac{1}{2}$ (C) 0
(D) $\frac{1}{2}$ (E) $\frac{\sqrt{3}}{2}$

3. $\lim_{h \rightarrow 0} \frac{(x+h)^3 - (x^3)}{h} =$

- (A) $-x^3$ (B) $-3x^2$ (C) $3x^2$
(D) x^3 (E) The limit does not exist



4. The graph of $y = f(x)$ is shown above. $\lim_{x \rightarrow 2} \left((f(x))^3 - 3f(x) + 7 \right) =$
- (A) 1 (B) 5 (C) 7 (D) 9 (E) Does not exist
5. If $f(x) = \begin{cases} \frac{x^2 - 3x - 4}{x + 1}, & x \neq -1 \\ 2, & x = -1 \end{cases}$, what is $\lim_{x \rightarrow -1} f(x)$?
- (A) -5 (B) 0 (C) 2 (D) 3 (E) Does not exist
6. $\lim_{x \rightarrow \infty} \left(\frac{2x^6 - 5x^3 + 10}{20 - 4x^2 - x^6} \right) =$
- (A) -2 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 2 (E) Does not exist
7. $\lim_{x \rightarrow \infty} \left(\frac{2x^5 - 5x^3 + 10}{20 - 4x^2 - x^6} \right) =$
- (A) -2 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 2

8. $\lim_{x \rightarrow \infty} \left(1 + e^{\frac{1}{2} + \frac{1}{x}} \right) =$

(A) $-\infty$ (B) 0 (C) $e^{\frac{1}{2}}$

(D) $1 + e^{\frac{1}{2}}$ (E) ∞

9. $\lim_{x \rightarrow 3^+} \frac{5}{3 - x} =$

(A) $-\infty$ (B) -5 (C) 0

(D) $\frac{5}{3}$ (E) ∞

10. If $\lim_{n \rightarrow \infty} \left(\frac{5n^3}{20 - 3n - kn^3} \right) = \frac{1}{2}$, then $k =$

(A) -10 (B) -4 (C) $\frac{1}{4}$ (D) 4 (E) 10

11. Which of the following is/are true about the function g if $g(x) = \frac{(x-2)^2}{x^2 + x - 6}$?

- I. g is continuous at $x = 2$
- II. The graph of g has a vertical asymptote at $x = -3$
- III. The graph of g has a horizontal asymptote at $y = 0$

(A) I only (B) II only (C) III only (D) I and II only (E) II and III only

$$12. f(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x > \frac{\pi}{4} \\ \tan x, & x = \frac{\pi}{4} \end{cases}$$

What is $\lim_{x \rightarrow \frac{\pi}{4}} f(x)$?

- (A) $-\infty$ (B) 0 (C) 1 (D) $\frac{\sqrt{2}}{2}$ (E) ∞

$$13. \lim_{x \rightarrow a} \left(\frac{\sqrt{x} - \sqrt{a}}{x - a} \right) =$$

- (A) $\frac{1}{2\sqrt{a}}$ (B) $\frac{1}{\sqrt{a}}$ (C) $\sqrt{a}$ (D) $2\sqrt{a}$ (E) Does not exist

$$14. \lim_{x \rightarrow 0^+} \frac{\ln 2x}{2x} =$$

- (A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

$$15. \text{ At } x = 4, \text{ the function given by } h(x) = \begin{cases} x^2, & x \leq 4 \\ 4x, & x > 4 \end{cases} \text{ is}$$

- (A) Undefined
(B) Continuous but not differentiable
(C) Differentiable but not continuous
(D) Neither continuous nor differentiable
(E) Both continuous and differentiable

Free Response 1

Let h be the function defined by the following:

$$h(x) = \begin{cases} |x-1| + 3, & 1 \leq x \leq 2 \\ ax^2 - bx, & x > 2 \end{cases}$$

a and b are constants.

(a) If $a = -1$ and $b = -4$, is $h(x)$ continuous for all x in $[1, \infty]$? Justify your answer.

(b) Describe all values of a and b such that h is a continuous function over the interval $[1, \infty]$.

(c) The function h will be continuous and differentiable over the interval $[1, \infty]$ for which values of a and b ?

Free Response 2 (No calculator)

Given the function $f(x) = \frac{x^3 + 2x^2 - 3x}{3x^2 + 3x - 6}$.

- (a) What are the zeros of $f(x)$?
- (b) What are the vertical asymptotes of $f(x)$?
- (c) The end behavior model of $f(x)$ is the function $g(x)$. What is $g(x)$?
- (d) What is $\lim_{x \rightarrow \infty} f(x)$? What is $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$?



Key

Page 5: 1, 2, does not exist

Practice Problems:

1. 0

2. $\frac{3}{5}$

3. ∞

4. $-\frac{1}{2}$

5. 0

6. $-\frac{1}{2}$

7. $\frac{1}{2}$

8. 7

9. 4

10. $-\infty$

11. ∞

12. does not exist

13. $-\infty$

14. $\frac{2\sqrt{2}}{\pi}$

15. $\frac{4}{\pi}$



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Multiple Choice Questions:

1. B
2. A
3. C
4. B
5. A
6. A
7. C
8. D
9. A
10. A
11. B
12. D
13. A
14. A
15. B

Free Response 1

Let h be the function defined by the following:

$$h(x) = \begin{cases} |x-1| + 3, & 1 \leq x \leq 2 \\ ax^2 - bx, & x > 2 \end{cases}$$

a and b are constants.

- (a) If $a = -1$ and $b = -4$, is $h(x)$ continuous for all x in $[1, \infty]$? Justify your answer.
- (b) Describe all values of a and b such that h is a continuous function over the interval $[1, \infty]$.
- (c) The function h will be continuous and differentiable over the interval $[1, \infty]$ for which values of a and b ?

(a)

$$|2-1| + 3 = 4a - 2b$$

$$4 = 4a - 2b$$

$$4 = 4(-1) - 2(-4)$$

$$4 = 4$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 4$$

$$f(2) = 4$$

The function, $h(x)$, is continuous for all x , given the a and b , because the function has a limit as x approaches 2, the function has a value as x approaches 2, and the limit is equal to the value.

(b)

$$4a - 2b = 4$$

$$4a = 4 + 2b$$

$$a = 1 + \frac{1}{2}b$$

The function is continuous for all

$$a = 1 + \frac{1}{2}b.$$

(c)

$$4 = 4a - 2b$$

$$1 = 4a - b$$

1 pt equation with substitutions

1 pt limits equal; 1 pt value

2 pts for finding a in terms of b

1 pt for continuity equation

1 pt for differentiability equation

Continued on next page.



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$$-b = 3$$

$$b = -3$$

$$a = -\frac{1}{2}$$

The function, $h(x)$, is continuous and differentiable when

$$a = -\frac{1}{2}$$

$$b = -3$$

1 pt for a ; 1 pt for b

Free Response 2 (No calculator)

Given the function $f(x) = \frac{x^3 + 2x^2 - 3x}{3x^2 + 3x - 6}$.

- What are the zeros of $f(x)$?
- What are the vertical asymptotes of $f(x)$?
- The end behavior model of $f(x)$ is the function $g(x)$. What is $g(x)$?
- What is $\lim_{x \rightarrow \infty} f(x)$? What is $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$?

(a) The zeros of the function, $f(x)$, occur at $x = -3, 0, 1$

3 pts, 1 for each zero

(b) There is a vertical asymptote at $x = -2$

1 pt for the vertical asymptote

(c) $g(x) = \frac{1}{3}x$

2 pts for $g(x)$

(d) $\lim_{x \rightarrow \infty} f(x) = \infty$

1 pt $\lim_{x \rightarrow \infty} f(x)$

$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$

2 pts for $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$