

Section 2.3

Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

In Section 2.2 you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

NOTE A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products involving three or more factors.

Proof Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned} \frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} + g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} f(x + \Delta x) \cdot \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

Note that $\lim_{\Delta x \rightarrow 0} f(x + \Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

The Product Rule can be extended to cover products involving more than two factors. For example, if f , g , and h are differentiable functions of x , then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

For instance, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned} \frac{dy}{dx} &= 2x \sin x \cos x + x^2 \cos x \cos x + x^2 \sin x (-\sin x) \\ &= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x). \end{aligned}$$

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx dy$ (as being negligible) and obtained the differential form $x dy + y dx$. This derivation resulted in the traditional form of the Product Rule.

(Source: *The History of Mathematics* by David M. Burton)

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned}
 h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}} \overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}} \overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}} && \text{Apply Product Rule.} \\
 &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\
 &= (12x - 8x^2) + (15 - 8x - 16x^2) \\
 &= -24x^2 + 4x + 15
 \end{aligned}$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned}
 D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x] \\
 &= -24x^2 + 4x + 15.
 \end{aligned}$$

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\begin{aligned}
 \frac{d}{dx}[3x^2 \sin x] &= 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2] && \text{Apply Product Rule.} \\
 &= 3x^2 \cos x + (\sin x)(6x) \\
 &= 3x^2 \cos x + 6x \sin x \\
 &= 3x(x \cos x + 2 \sin x)
 \end{aligned}$$

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned}
 \frac{dy}{dx} &= \overbrace{(2x) \left(\frac{d}{dx}[\cos x] \right) + (\cos x) \left(\frac{d}{dx}[2x] \right)}^{\text{Product Rule}} - \overbrace{2 \frac{d}{dx}[\sin x]}^{\text{Constant Multiple Rule}} \\
 &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x) \\
 &= -2x \sin x
 \end{aligned}$$

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

The Quotient Rule

THEOREM 2.8 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= \lim_{\Delta x \rightarrow 0} \frac{\frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \rightarrow 0} \frac{g(x)f(x + \Delta x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{g(x)f(x + \Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)} \\ &= \frac{\lim_{\Delta x \rightarrow 0} \frac{g(x)[f(x + \Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \rightarrow 0} \frac{f(x)[g(x + \Delta x) - g(x)]}{\Delta x}}{\lim_{\Delta x \rightarrow 0} [g(x)g(x + \Delta x)]} \\ &= \frac{g(x) \left[\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] - f(x) \left[\lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]}{\lim_{\Delta x \rightarrow 0} [g(x)g(x + \Delta x)]} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned}$$

Note that $\lim_{\Delta x \rightarrow 0} g(x + \Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

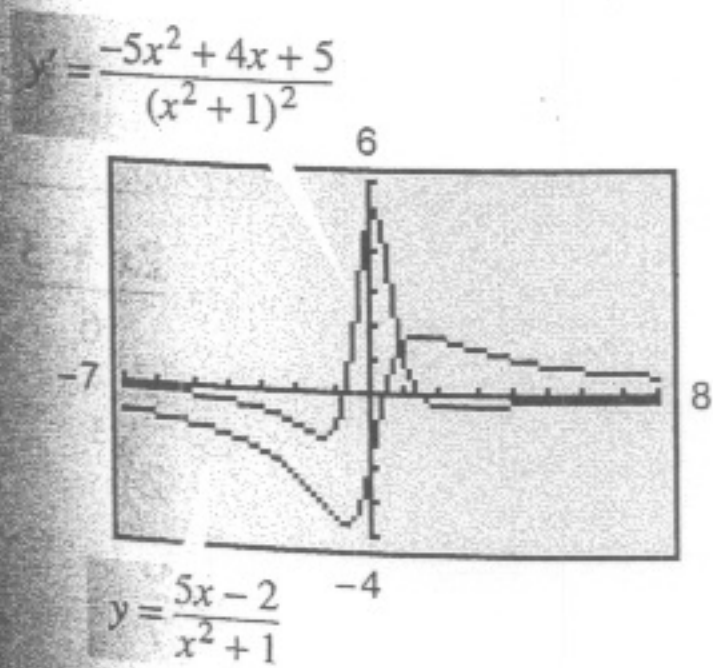
EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \frac{5x - 2}{x^2 + 1}$.

Solution

$$\begin{aligned} \frac{d}{dx} \left[\frac{5x - 2}{x^2 + 1} \right] &= \frac{(x^2 + 1) \frac{d}{dx} [5x - 2] - (5x - 2) \frac{d}{dx} [x^2 + 1]}{(x^2 + 1)^2} && \text{Apply Quotient Rule.} \\ &= \frac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2} \\ &= \frac{(5x^2 + 5) - (10x^2 - 4x)}{(x^2 + 1)^2} \\ &= \frac{-5x^2 + 4x + 5}{(x^2 + 1)^2} \end{aligned}$$

TECHNOLOGY A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative
Figure 2.22

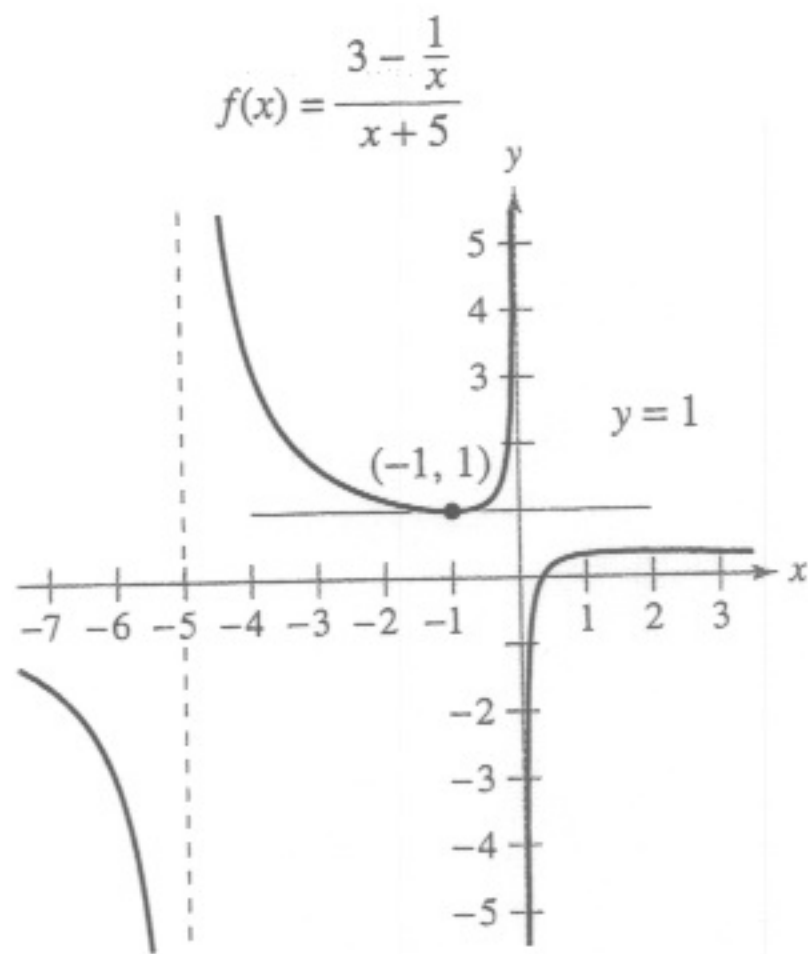
Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \frac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
Figure 2.23

$$\begin{aligned}
 f(x) &= \frac{3 - (1/x)}{x + 5} && \text{Write original function.} \\
 &= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} && \text{Multiply numerator and denominator by } x. \\
 &= \frac{3x - 1}{x^2 + 5x} && \text{Rewrite.} \\
 f'(x) &= \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} && \text{Quotient Rule} \\
 &= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2} \\
 &= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} && \text{Simplify.}
 \end{aligned}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23.

Not every quotient needs to be differentiated by the Quotient Rule. For example, each quotient in the next example can be considered as the product of a constant times a function of x . In such cases it is more convenient to use the Constant Multiple Rule.

EXAMPLE 6 Using the Constant Multiple Rule

Original Function	Rewrite	Differentiate	Simplify
a. $y = \frac{x^2 + 3x}{6}$	$y = \frac{1}{6}(x^2 + 3x)$	$y' = \frac{1}{6}(2x + 3)$	$y' = \frac{2x + 3}{6}$
b. $y = \frac{5x^4}{8}$	$y = \frac{5}{8}x^4$	$y' = \frac{5}{8}(4x^3)$	$y' = \frac{5}{2}x^3$
c. $y = \frac{-3(3x - 2x^2)}{7x}$	$y = -\frac{3}{7}(3 - 2x)$	$y' = -\frac{3}{7}(-2)$	$y' = \frac{6}{7}$
d. $y = \frac{9}{5x^2}$	$y = \frac{9}{5}(x^{-2})$	$y' = \frac{9}{5}(-2x^{-3})$	$y' = -\frac{18}{5x^3}$

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.