A24

MATHEMATICS

SUPPORT CENTRE

Title: Sigma Notation

Target: On completion of this worksheet you should understand what is meant by sigma notation and be able to evaluate simple expressions written in sigma notation.

It would be useful to have a short hand way of writing such expressions as 1+2+3+4+5+6+7+8
We can use the symbol Σ (sigma) which means 'sum of'. We write

$$1+2+3+4+5+6+7+8=\sum_{r=1}^{8} r$$

r takes integer values from 1 to 8 so $\sum_{r=1}^{8} r = 36$

Exercise

Evaluate the following:

1. $\sum_{r=1}^{4} r$ 2. $\sum_{r=2}^{7} r$

3. $\sum_{i=0}^{3} i$ 4. $\sum_{k=5}^{11} k$

(Answers: 10, 27, 6, 56)

The expression after the Σ can more complicated than just one letter.

Examples

1. $\sum_{r=1}^{4} r^2 = 1^2+2^2+3^2+4^2 = 30$

2. $\sum_{k=2}^{4} k(k-1) = 2\times(2-1)+3\times(3-1)+4\times(4-1)$

$$= 2\times1+3\times2+4\times3$$
$$= 20$$

3. $\sum_{r=1}^{5} (r^3+2) = (1^3+2)+(2^3+2)+(3^3+2)+$

$$(4^3+2)+(5^3+2)$$
$$= 3+10+29+66+127$$
$$= 235$$

Exercise

Evaluate the following:

1. $\sum_{r=11}^{14} r^2$ 2. $\sum_{n=3}^{7} n(n+2)$

2. $\sum_{i=1}^{5} 3i$ 4. $\sum_{k=0}^{4} \frac{k^4}{2}$

5. $\sum_{r=2}^{6} \frac{1}{2} r(r+3)$ 6. $\sum_{k=1}^{4} (k+1)(k+2)$

(Answers: 630, 185, 45, 177, 75, 68)

This notation can be used in general cases:

$$\sum_{r=1}^{n} r^4 = 1^4+2^4+3^4+4^4+\cdots+n^4$$

Examples

Write down the first 3 terms and the last term of the following series:

1. $\sum_{r=2}^{n} r(r-5) = 2(2-5)+3(3-5)+4(4-5)+\cdots$

$$= 2\times(-3)+3\times(-2)+4\times(-1)+\cdots$$
$$= (-6)+(-6)+(-4)+\cdots$$

The last term $= n(n-5)$

2. $\sum_{k=5}^{n} \frac{1}{k} = \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots+\frac{1}{n}$

3. $\sum_{w=0}^{n} \frac{w}{w+1} = \frac{0}{0+1}+\frac{1}{1+1}+\frac{2}{2+1}+\cdots+\frac{n}{n+1}$

$$= 0+\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}$$

Note: The last term is always the same as the general term but with a change of letter.

Exercise

Write down the first 3 terms and the last term of the following series:

1. $\sum_{r=3}^{n} r^2(r-1)$
2. $\sum_{k=6}^{n} (2k+4)$
3. $\sum_{k=1}^{n} \frac{2}{k+1}$
4. $\sum_{r=2}^{n} \frac{r}{r+2}$
5. $\sum_{k=1}^{n} (2^k - 1)$
6. $\sum_{r=1}^{n} (-1)^r r^2$

Answers:

1. $18+48+75+\cdots+n^2(n-1)$
2. $16+18+20+\cdots+(2n+4)$
3. $1+\frac{2}{3}+\frac{1}{2}+\cdots+\frac{2}{n+1}$
4. $\frac{1}{2}+\frac{3}{5}+\frac{2}{3}+\cdots+\frac{n}{n+2}$
5. $1+3+7+\cdots+(2^n-1)$
6. $(-1)+4+(-9)+\cdots+(-1)^n n^2$

Note: In the this example the terms alternate in sign depending on whether r is positive or negative.

Further Examples

1. $\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$
2. $\sum_{k=1}^{n} y_k = y_1 + y_2 + y_3 + y_4 + \cdots + y_n$

If $x_1 = 2,\ x_2 = 5,\ x_3 = 1,\ x_4 = 3,\ y_1 = 4,\ y_2 = 1$ $y_3 = 6$ and $y_4 = 2$ evaluate the following :

3. $\sum_{i=1}^{4} x_i = 2+5+1+3 = 11$
4. $\sum_{i=1}^{3} x_i^2 = 2^2 + 5^2 + 1^2 = 30$
5. $\sum_{k=1}^{4} x_k y_k = 2\times4+5\times1+1\times6+3\times2 = 25$
6. $\sum_{i=1}^{3} y_{i+1} = y_2 + y_3 + y_4 = 1+6+2 = 9$

Exercise

Evaluate the following if $x_1 = 2,\ x_2 = 4,\ x_3 = -1$ $x_4 = 5,\ y_1 = 1,\ y_2 = 6,\ y_3 = 3$ and $y_4 = -2$

1. $\sum_{i=1}^{4} x_i$
2. $\sum_{i=1}^{4} y_i$
3. $\sum_{i=2}^{4} x_i y_i$
4. $\sum_{k=1}^{2} y_k^2$
5. $\sum_{j=2}^{4} x_{j-1} y_j$
6. $\sum_{i=1}^{3} x_i x_{i+1}$
7. $\sum_{i=1}^{4} (y_i^3 - 1)$
8. $\sum_{i=1}^{4} x_i \sum_{j=1}^{2} y_j$

(Answers: 10, 8, 11, 37, 26, -1, 232, 70)