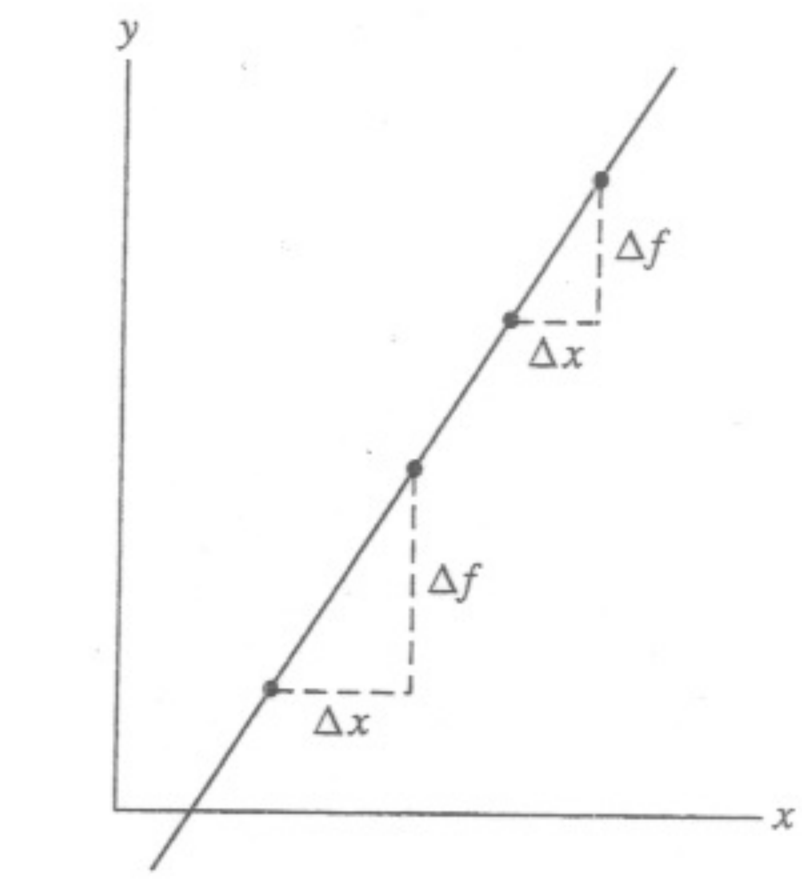


FIGURE 7 For a linear function $f(x) = mx + b$, the ratio $\Delta f/\Delta x$ is equal to the slope m for every interval.

To conclude this section, we recall an important point discussed in Section 1.2: *For any linear function $f(x) = mx + b$, the average rate of change over every interval is equal to the slope m* (Figure 7). We verify as follows:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{(mx_1 + b) - (mx_0 + b)}{x_1 - x_0} = \frac{m(x_1 - x_0)}{x_1 - x_0} = m$$

The instantaneous rate of change at $x = x_0$, which is the limit of these average rates, is also equal to m. This makes sense graphically because all secant lines and all tangent lines to the graph of $f(x)$ coincide with the graph itself.

2.1 SUMMARY

- The *average rate of change* of $y = f(x)$ over an interval $[x_0, x_1]$:

$$\text{Average rate of change} = \frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad (x_1 \neq x_0)$$

- The *instantaneous rate of change* is the limit of the average rates of change.
- Graphical interpretation:
  - Average rate of change is the slope of the secant line through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the graph of $f(x)$.
  - Instantaneous rate of change is the slope of the tangent line at x_0.
- To estimate the instantaneous rate of change at $x = x_0$, compute the average rate of change over several intervals $[x_0, x_1]$ (or $[x_1, x_0]$) for x_1 close to x_0.
- The velocity of an object in linear motion is the rate of change of position $s(t)$.
- Linear function $f(x) = mx + b$: The average rate of change over every interval and the instantaneous rate of change at every point are equal to the slope m.

2.1 EXERCISES

Preliminary Questions

1. Average velocity is equal to the slope of a secant line through two points on a graph. Which graph?

2. Can instantaneous velocity be defined as a ratio? If not, how is instantaneous velocity computed?

3. What is the graphical interpretation of instantaneous velocity at a moment $t = t_0$?

4. What is the graphical interpretation of the following statement? The average rate of change approaches the instantaneous rate of change as the interval $[x_0, x_1]$ shrinks to x_0.

5. The rate of change of atmospheric temperature with respect to altitude is equal to the slope of the tangent line to a graph. Which graph? What are possible units for this rate?

Exercises

1. A ball dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds.

(a) How far does the ball travel during the time interval [2, 2.5]?

(b) Compute the average velocity over [2, 2.5].

(c) Compute the average velocity for the time intervals in the table and estimate the ball's instantaneous velocity at $t = 2$.

Interval	[2, 2.01]	[2, 2.005]	[2, 2.001]	[2, 2.00001]
Average velocity				

2. A wrench released from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds. Estimate the instantaneous velocity at $t = 3$.

3. Let $v = 20\sqrt{T}$ as in Example 2. Estimate the instantaneous rate of change of v with respect to T when $T = 300$ K.

4. Compute $\Delta y/\Delta x$ for the interval [2, 5], where $y = 4x - 9$. What is the instantaneous rate of change of y with respect to x at $x = 2$?

In Exercises 5–6, a stone is tossed vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $h(t) = 15t - 4.9t^2$ m.

5. Compute the stone's average velocity over the time interval [0.5, 2.5] and indicate the corresponding secant line on a sketch of the graph of $h(t)$.

6. Compute the stone's average velocity over the time intervals [1, 1.01], [1, 1.001], [1, 1.0001] and [0.99, 1], [0.999, 1], [0.9999, 1], and then estimate the instantaneous velocity at $t = 1$.

7. With an initial deposit of \$100, the balance in a bank account after t years is $f(t) = 100(1.08)^t$ dollars.

(a) What are the units of the rate of change of $f(t)$?

(b) Find the average rate of change over [0, 0.5] and [0, 1].

(c) Estimate the instantaneous rate of change at $t = 0.5$ by computing the average rate of change over intervals to the left and right of $t = 0.5$.

8. The position of a particle at time t is $s(t) = t^3 + t$. Compute the average velocity over the time interval [1, 4] and estimate the instantaneous velocity at $t = 1$.

9. Figure 8 shows the estimated number N of Internet users in Chile, based on data from the United Nations Statistics Division.

(a) Estimate the rate of change of N at $t = 2003.5$.

(b) Does the rate of change increase or decrease as t increases? Explain graphically.

(c) Let R be the average rate of change over [2001, 2005]. Compute R.

(d) Is the rate of change at $t = 2002$ greater than or less than the average rate R? Explain graphically.

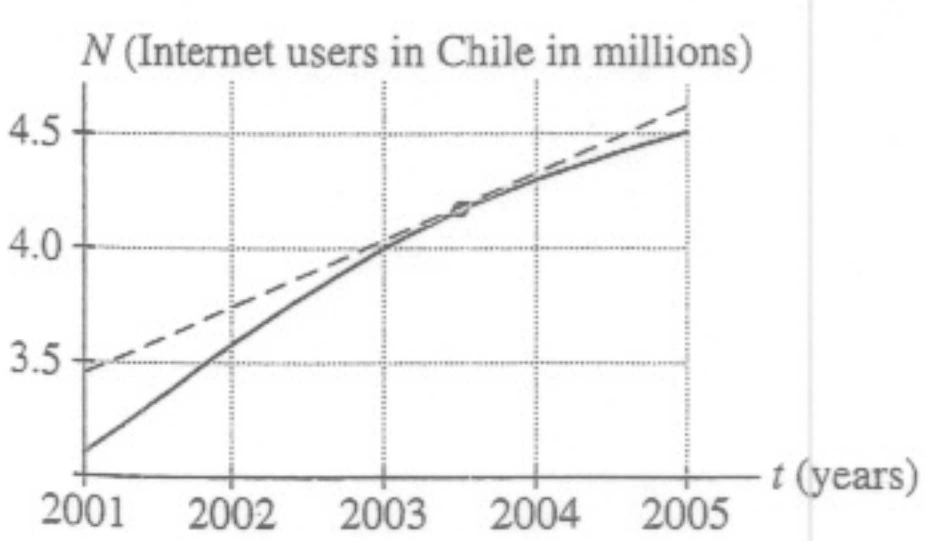


FIGURE 8

10. The **atmospheric temperature** T (in °C) at altitude h meters above a certain point on earth is $T = 15 - 0.0065h$ for $h \le 12{,}000$ m. What are the average and instantaneous rates of change of T with respect to h? Why are they the same? Sketch the graph of T for $h \le 12{,}000$.

In Exercises 11–18, estimate the instantaneous rate of change at the point indicated.

11. $P(x) = 3x^2 - 5$; $x = 2$

12. $f(t) = 12t - 7$; $t = -4$

13. $y(x) = \dfrac{1}{x+2}$; $x = 2$

14. $y(t) = \sqrt{3t+1}$; $t = 1$

15. $f(x) = e^x$; $x = 0$

16. $f(x) = e^x$; $x = e$

17. $f(x) = \ln x$; $x = 3$

18. $f(x) = \tan^{-1} x$; $x = \frac{\pi}{4}$

19. The height (in centimeters) at time t (in seconds) of a small mass oscillating at the end of a spring is $h(t) = 8\cos(12\pi t)$.

(a) Calculate the mass's average velocity over the time intervals [0, 0.1] and [3, 3.5].

(b) Estimate its instantaneous velocity at $t = 3$.

20. The number $P(t)$ of *E. coli* cells at time t (hours) in a petri dish is plotted in Figure 9.

(a) Calculate the average rate of change of $P(t)$ over the time interval [1, 3] and draw the corresponding secant line.

(b) Estimate the slope m of the line in Figure 9. What does m represent?

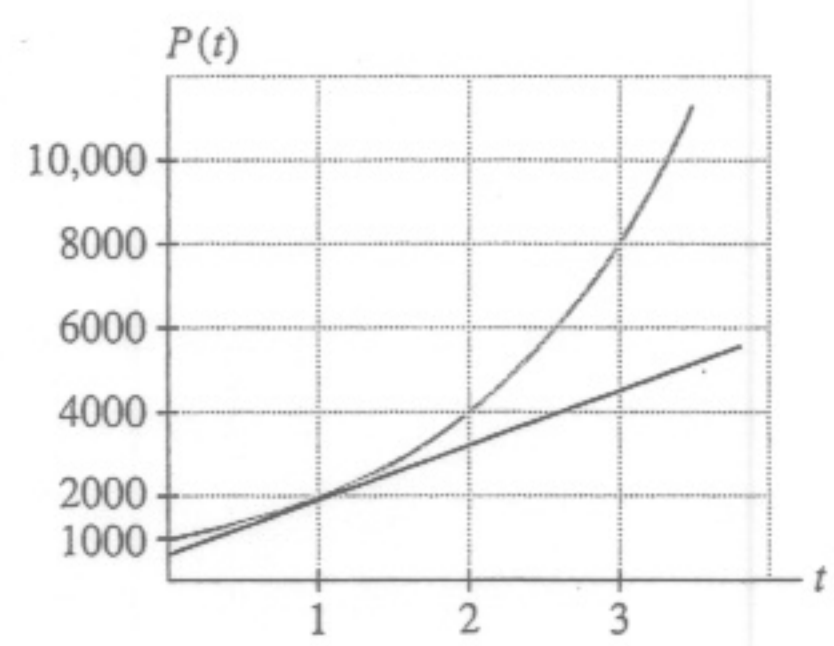


FIGURE 9 Number of *E. coli* cells at time t.

21. Assume that the period T (in seconds) of a pendulum (the time required for a complete back-and-forth cycle) is $T = \frac{3}{2}\sqrt{L}$, where L is the pendulum's length (in meters).

(a) What are the units for the rate of change of T with respect to L? Explain what this rate measures.

(b) Which quantities are represented by the slopes of lines A and B in Figure 10?

(c) Estimate the instantaneous rate of change of T with respect to L when $L = 3$ m.

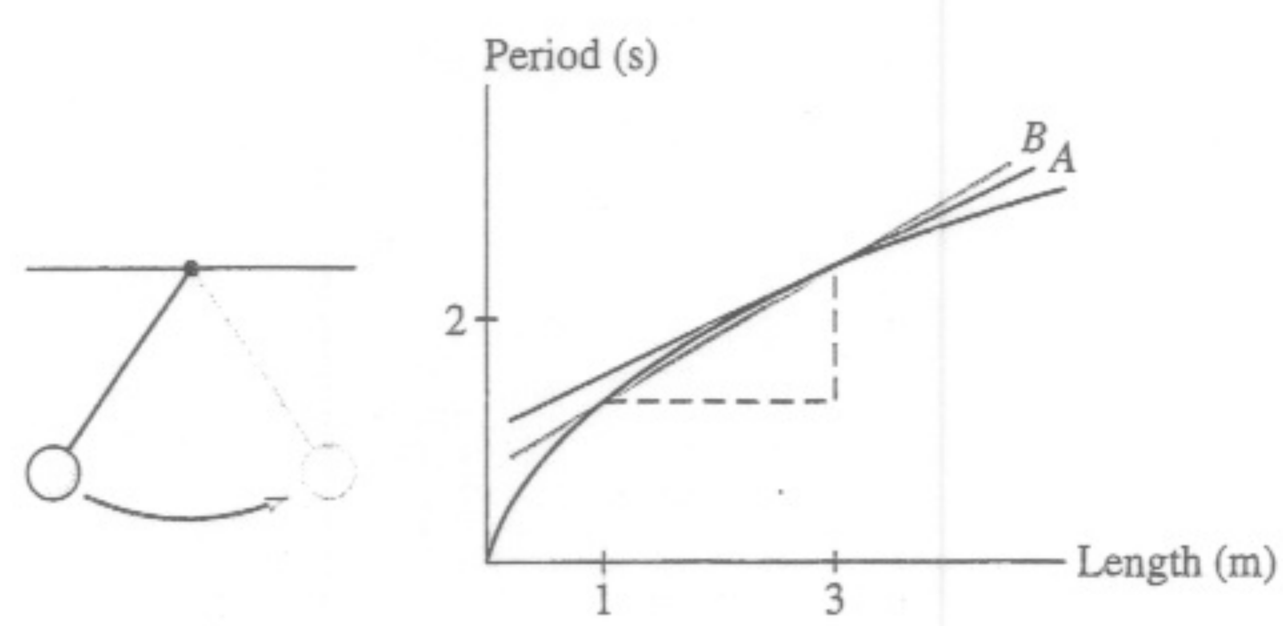


FIGURE 10 The period T is the time required for a pendulum to swing back and forth.

22. The graphs in Figure 11 represent the positions of moving particles as functions of time.

(a) Do the instantaneous velocities at times t_1, t_2, t_3 in (A) form an increasing or a decreasing sequence?

(b) Is the particle speeding up or slowing down in (A)?

(c) Is the particle speeding up or slowing down in (B)?

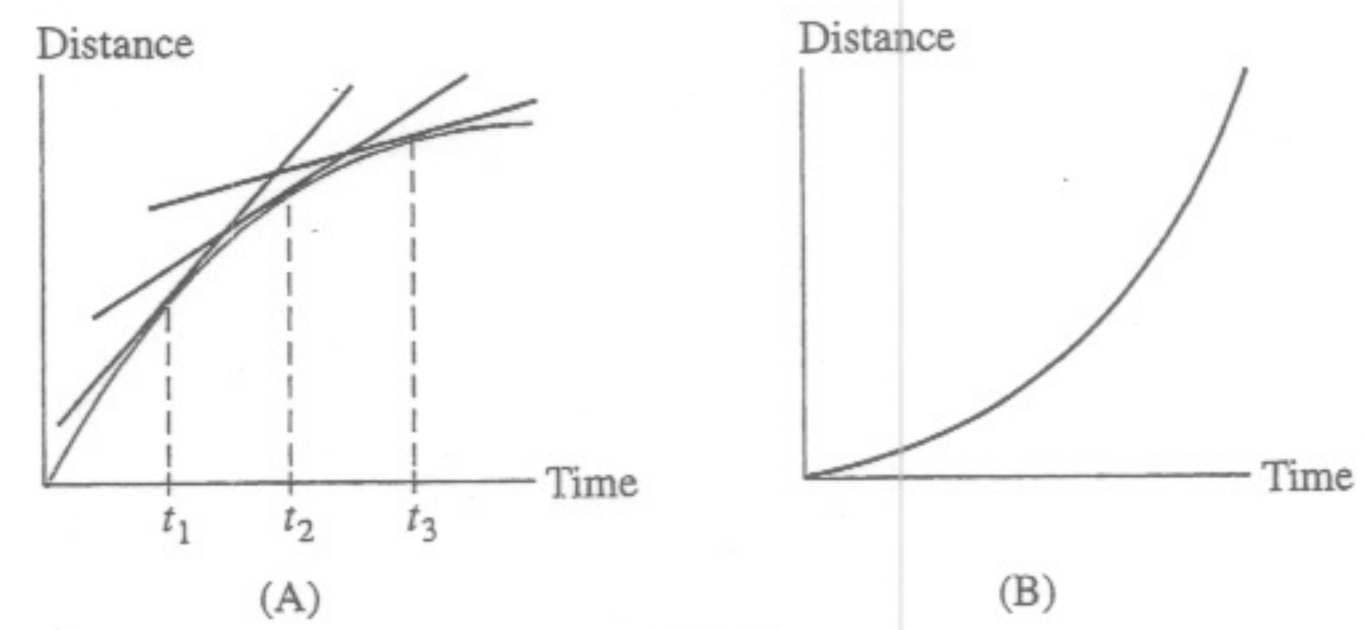


FIGURE 11

23. GU An advertising campaign boosted sales of Crunchy Crust frozen pizza to a peak level of S_0 dollars per month. A marketing study showed that after t months, monthly sales declined to

$$S(t) = S_0 g(t), \quad \text{where } g(t) = \frac{1}{\sqrt{1+t}}.$$

Do sales decline more slowly or more rapidly as time increases? Answer by referring to a sketch of the graph of $g(t)$ together with several tangent lines.

24. The fraction of a city's population infected by a flu virus is plotted as a function of time (in weeks) in Figure 12.

(a) Which quantities are represented by the slopes of lines A and B? Estimate these slopes.

(b) Is the flu spreading more rapidly at $t = 1, 2,$ or 3?

(c) Is the flu spreading more rapidly at $t = 4, 5,$ or 6?

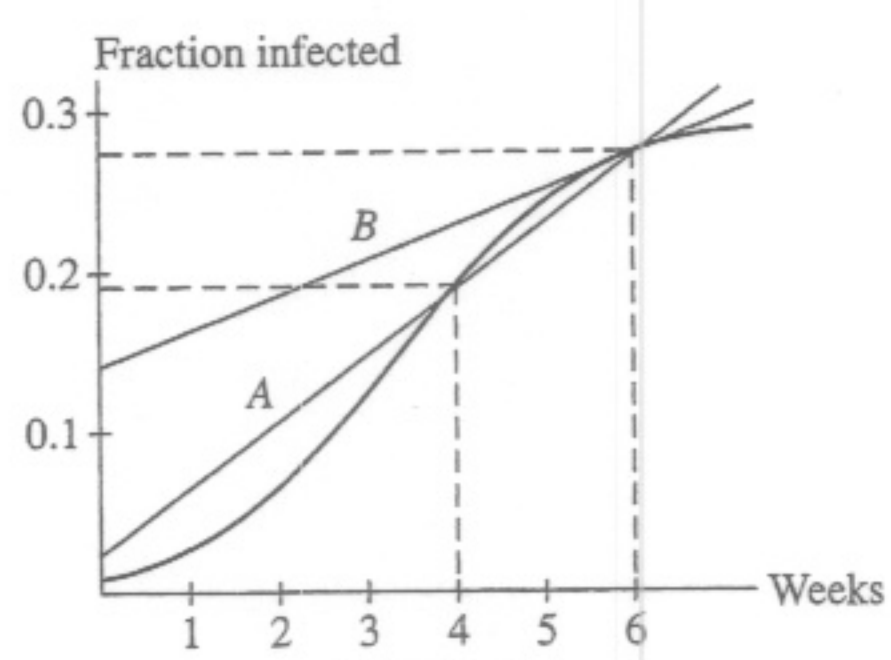


FIGURE 12

25. The graphs in Figure 13 represent the positions s of moving particles as functions of time t. Match each graph with a description:

(a) Speeding up

(b) Speeding up and then slowing down

(c) Slowing down

(d) Slowing down and then speeding up

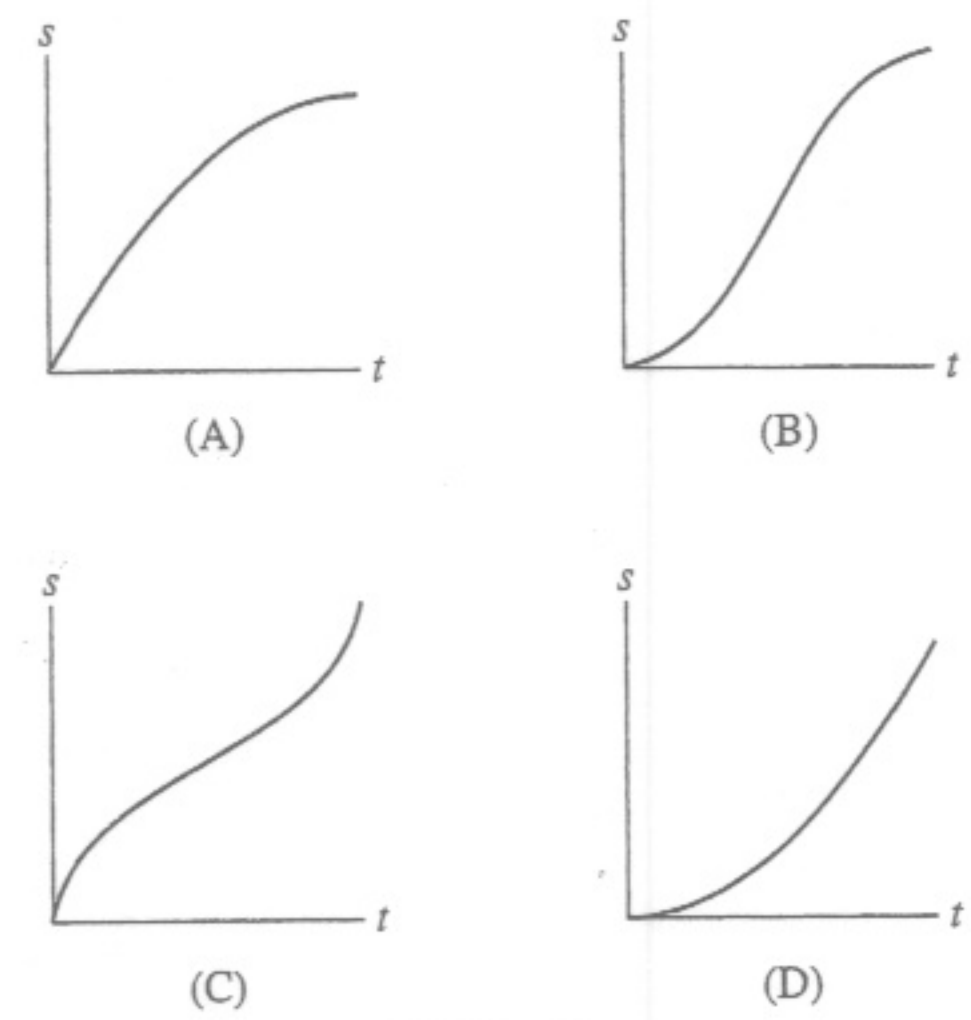


FIGURE 13

26. An epidemiologist finds that the percentage $N(t)$ of susceptible children who were infected on day t during the first three weeks of a measles outbreak is given, to a reasonable approximation, by the formula (Figure 14)

$$N(t) = \frac{100t^2}{t^3 + 5t^2 - 100t + 380}$$

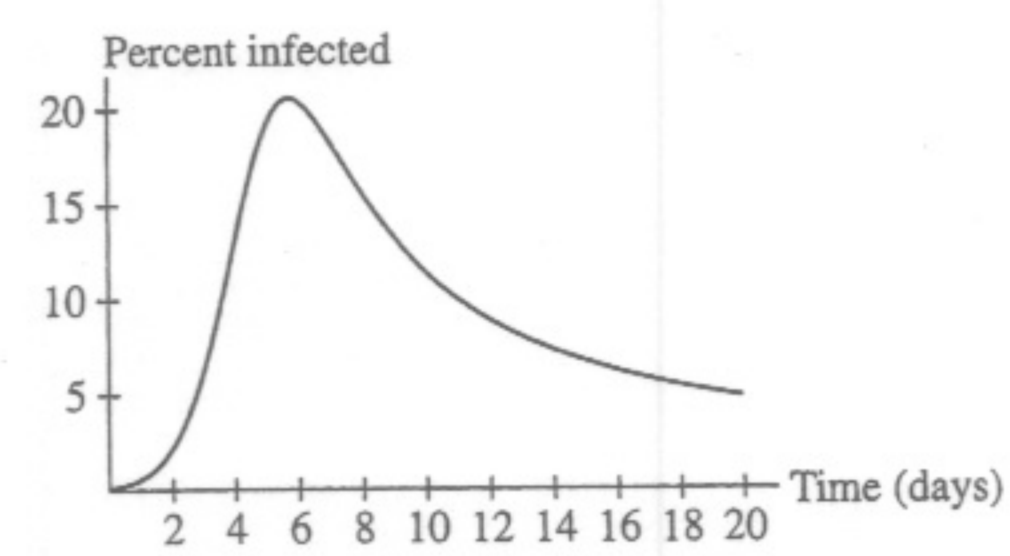


FIGURE 14 Graph of $N(t)$.

(a) Draw the secant line whose slope is the average rate of change in infected children over the intervals [4, 6] and [12, 14]. Then compute these average rates (in units of percent per day).

(b) Is the rate of decline greater at $t = 8$ or $t = 16$?

(c) Estimate the rate of change of $N(t)$ on day 12.

27. The fungus *Fusarium exosporium* infects a field of flax plants through the roots and causes the plants to wilt. Eventually, the entire field is infected. The percentage $f(t)$ of infected plants as a function of time t (in days) since planting is shown in Figure 15.

(a) What are the units of the rate of change of $f(t)$ with respect to t? What does this rate measure?

(b) Use the graph to rank (from smallest to largest) the average infection rates over the intervals [0, 12], [20, 32], and [40, 52].

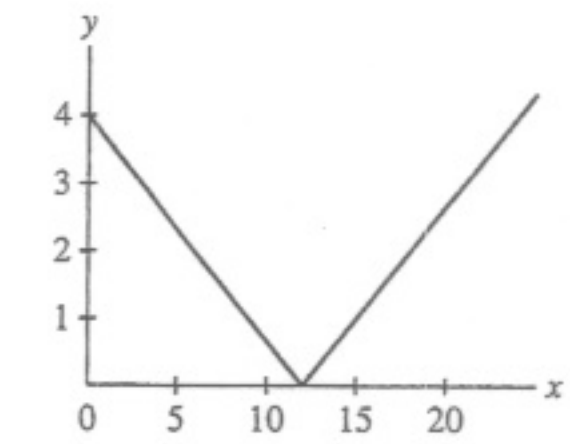
(c) Use the following table to compute the average rates of infection over the intervals [30, 40], [40, 50], [30, 50].

Days	0	10	20	30	40	50	60
Percent infected	0	18	56	82	91	96	98

(d) Draw the tangent line at $t = 40$ and estimate its slope.

Key

35. Let $g(x) = f(\frac{1}{3}x)$. Then $g(x - 3b) = f\left(\frac{1}{3}(x - 3b)\right) = f\left(\frac{1}{3}x - b\right)$. The graph of $y = |\frac{1}{3}x - 4|$:



37. $f(t) = t^4$ and $g(t) = 12t + 9$ **39.** 4π

41. **(a)** $a = b = \pi/2$ **(b)** $a = \pi$

43. $x = \pi/2, x = 7\pi/6, x = 3\pi/2$ and $x = 11\pi/6$

45. There are no solutions

47. **(a)** No match. **(b)** No match. **(c)** (i) **(d)** (iii)

49. $f^{-1}(x) = \sqrt[3]{x^2 + 8}$; $D : \{x : x \geq 0\}$; $R : \{y : y \geq 2\}$

51. For $\{t : t \leq 3\}$, $h^{-1}(t) = 3 - \sqrt{t}$. For $t \geq 3$, $h^{-1}(t) = 3 + \sqrt{t}$.

53. **(a)** (iii) **(b)** (iv) **(c)** (ii) **(d)** (i)

Chapter 2

Section 2.1 Preliminary Questions

1. The graph of position as a function of time
2. No. Instantaneous velocity is defined as the limit of average velocity as time elapsed shrinks to zero.
3. The slope of the line tangent to the graph of position as a function of time at $t = t_0$
4. The slope of the secant line over the interval $[x_0, x_1]$ approaches the slope of the tangent line at $x = x_0$.
5. The graph of atmospheric temperature as a function of altitude. Possible units for this rate of change are °F/ft or °C/m.

Section 2.1 Exercises

1. **(a)** 11.025 m **(b)** 22.05 m/s

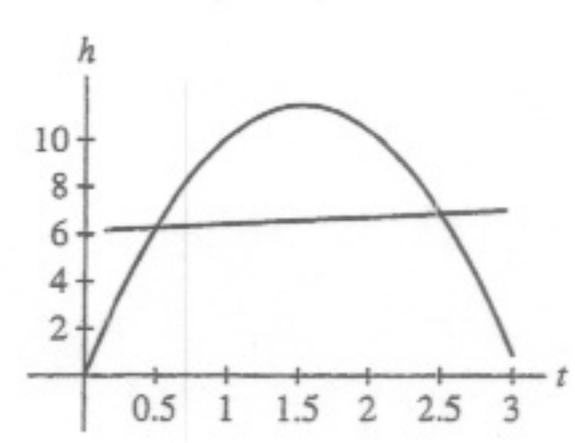
(c)

time interval	[2, 2.01]	[2, 2.005]	[2, 2.001]	[2, 2.00001]
average velocity	19.649	19.6245	19.6049	19.600049

The instantaneous velocity at $t = 2$ is 19.6 m/s.

3. 0.57735 m/(s · K)

5. 0.3 m/s



7. **(a)** Dollars/year **(b)** [0, 0.5]: 7.8461; [0, 1]: 8
(c) Approximately $8/yr

9. **(a)** Approximately 0.283 million Internet users per year.
(b) Decreases
(c) Approximately 0.225 million Internet users per year.
(d) Greater than

11. 12 **13.** −0.06 **15.** 1.00 **17.** 0.333

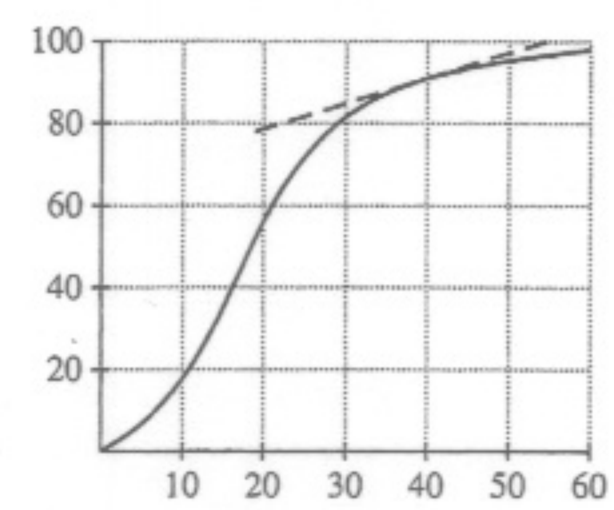
19. **(a)** [0, 0.1]: −144.721 cm/s; [3, 3.5]: 0 cm/s **(b)** 0 cm/s

21. **(a)** Seconds per meter; measures the sensitivity of the period of the pendulum to a change in the length of the pendulum.
(b) B: average rate of change in T from $L = 1$ m to $L = 3$ m; A: instantaneous rate of change of T at $L = 3$ m.
(c) 0.4330 s/m.

23. Sales decline more slowly as time increases.

25.
- In graph (A), the particle is (c) slowing down.
- In graph (B), the particle is (b) speeding up and then slowing down.
- In graph (C), the particle is (d) slowing down and then speeding up.
- In graph (D), the particle is (a) speeding up.

27. **(a)** Percent /day; measures how quickly the population of flax plants is becoming infected.
(b) [40, 52], [0, 12], [20, 32]
(c) The average rates of infection over the intervals [30, 40], [40, 50], [30, 50] are .9, .5, .7 %/d, respectively.
(d) 0.55%/d



31. (B)

33. Interval $[1, t]$: average rate of change is $t + 1$; interval $[2, t]$: average rate of change is $t + 2$

35. $x^2 + 2x + 4$

Section 2.2 Preliminary Questions

1. 1 **2.** π **3.** 20 **4.** Yes

5. $\lim_{x\to 1-} f(x) = \infty$ and $\lim_{x \to 1+} f(x) = 3$

6. No **7.** Yes

Section 2.2 Exercises

1.

x	0.998	0.999	0.9995	0.99999
$f(x)$	1.498501	1.499250	1.499625	1.499993

x	1.00001	1.0005	1.001	1.002
$f(x)$	1.500008	1.500375	1.500750	1.501500

The limit as $x \to 1$ is $\frac{3}{2}$.