Optimization Problems - Classwork

Many times in life we are asked to do an optimization problem - that is, find the largest or smallest value of some quantity that will fufill a need. Typical situations are:

- find the route which will minimize the time it takes me to get to school.
- build a structure using the least amount of material.
- build a structure costing the least amount of money.
- build a yard enclosing the most amount of space.
- find the least medication one should take to help a medical problem.
- find how the most one should charge for a CD in order to make as much money as possible.

All of these situations have something in common - they are all trying to maximize or minimize some quantity. This lends itself to a calculus solution. We have spent the better part of last month trying to find maximum and minimum values of functions. In every optimization problem, you are always looking for a quantity to be maximimed or minimized. So in solving word problems, you must look carefully for certain words among all the verbiage. Look for words like "minimize area", "smallest volume", "least amount of time", "shortest distance", "cheapest price." On the following pages, there are a wealth of problems. Quickly examine each and underline the key words which tell you what kind of problem it is.

Methods for Solving Optimization Problems

1. Assign variables to all given quantities and quantities to be determined. Don't be afraid to use letters you usually do not use (*p*,*m*,*g*, etc.). When feasible, make a sketch of the problem.

2. Making a chart of possible answers allows you to see a relationship between variables. While not necessary, it is helpful.

3. Write a "primary" equation for the quantity you found that needs to be maximized or minimized

   Area of Rectangle = length • width — Hypotenuse = $\sqrt{x^2 + y^2}$

   Distance = rate • time — Perimeter of rectangle = 2• length + 2• width

   Volume of rectangular solid = length • width • height — Volume of cylinder = π(radius^2) • height

4. Reduce the right side of this "primary equation" to one having a single variable. If there is more than one variable on the right side, you must write a "secondary" equation (a restriction or constraint) relating the variables of the primary equation.

5. Take the derivative of the equation and **set equal to zero**. If you get more than one answer, make a sign chart to determine whether it represents a maximum or minimum. Pay attention to whether that value makes sense. Time is rarely negative (it can't take negative 7 hours to run a race). You cannot use more than you have (you can't have a length of 8 feet when you only have 6 feet of fencing).

6. **Be sure that you answer the question that is asked.** If you are asked to find a minimum or maximum value of some quantity, you must plug your answer from (4) into your primary equation.

7. If you are to find a maximum or minimum on a closed interval, you must test the endpoints as well. Make sure your work is clear.

8. You can verify your answers by graphing your primary equation with one variable on the calculator. Use your [2nd] [CALC] maximum or minimum function.

Example 1) Two numbers add up to 40. Find the numbers that maximize their product.

Smaller Number						
Larger Number						
Product						

Primary *Secondary*

Example 2) A rectangle has a perimeter of 71 feet. What length and width should it have so that its area is a maximum? What is this maximum area?

Width						
Length						
Area						

Primary *Secondary*

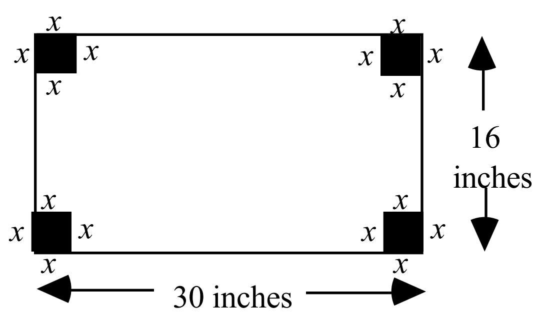
Example 3) Find two positive numbers that minimize the sum of twice the first number plus the second if the product of the two numbers is 288.

First Number						
Second Number						
Sum						

Primary *Secondary*

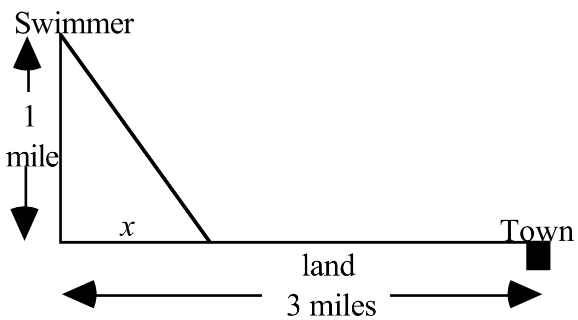
Example 4) An open box is to be made from a piece of metal 16 by 30 inches by cutting out squares of equal size from the corners and bending up the sides. What size square should be cut out to create a box with greatest volume. What is the maximum volume as well?

Primary



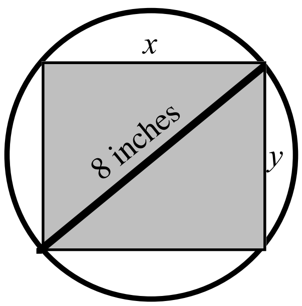
Example 5) I am 1 mile in the ocean and wish to get to a town 3 miles down the coast which is very rocky. I need to swim to the shore and then walk along the shore. What point should I swim to along the shoreline so that the time it takes to get to town is a minimum? I swim at 2 mph and walk at 4 mph.

Primary



Example 6) . Find the dimensions of the largest area rectangle which can be inscribed into a circle of radius 4 inches.

primary *secondary*



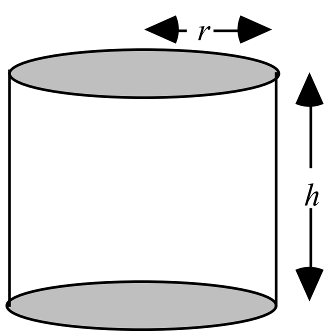
How would this problem change if the radius were *r* inches?

Example 7) A 6-oz. can of Friskies Cat food contains a volume of approximately 14.5 cubic inches. How should the can be constructed so that the material made to make the can is a minimum?

Primary *Secondary*

Surface Area = Area of side + Area of Top & bottom $V = \pi r^2 h$

$S = 2\pi rh + 2\pi r^2$



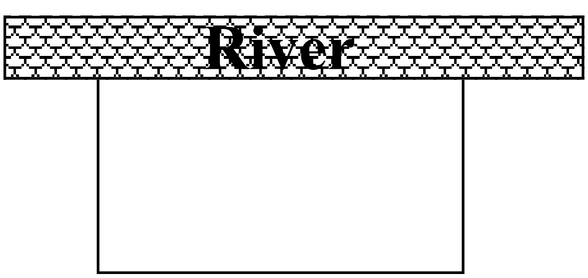
Optimization Problems - Homework

1. Find two numbers whose sum is 10 for which the sum of their squares is a minimum.

2. Find nonnegative numbers x and y whose sum is 75 and for which the value of xy^2 is as large as possible.

3. A ball is thrown straight up in the air from ground level. Its height after t seconds is given by $s(t) = -16t^2 + 50t$. When does the ball reach it maximum height? What is its maximum height?

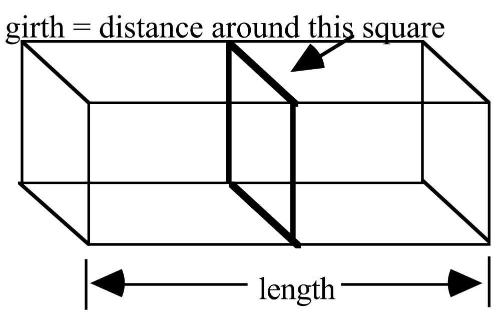
4. A farmer has 2,000 feet of fencing to enclose a pasture area. The field will be in the shape of a rectangle and will be placed against a river where there is no fencing needed. What is the largest area field that can be created and what are its dimensions?



5. A fisheries biologist is stocking fish in a lake. She knows that when there are n fish per unit of water, the average weight of each fish will be $W(n) = 500 - 2n$, measured in grams. What is the value of n that will maximize the total fish weight after one season. *Hint: Total Weight = number of fish • average weight of a fish.*

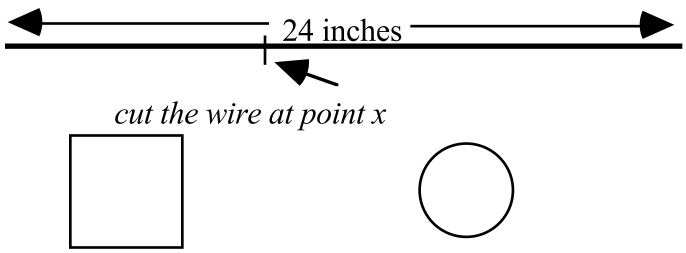
6. The size of a population of bacteria introduced to a food grows according to the formula $P(t) = \dfrac{6000t}{60 + t^2}$ where t is measured in weeks. Determine when the bacteria will reach its maximum size. What is the maximum size of the population?

7. The U.S. Postal Service will accept a box for domestic shipping only if the sum of the length and the girth (distance around) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that can be sent.



8. Blood pressure in a patient will drop by an amount $D(x)$ where $D(x) = 0.025x^2(30 - x)$ where x is the amount of drug injected in cm^3. Find the dosage that provides the greatest drop in blood pressure. What is the drop in blood pressure?

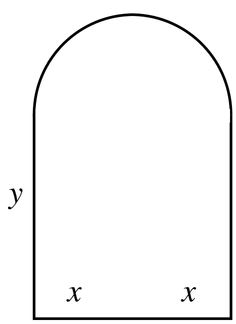
9. A wire 24 inches long is cut into two pieces. One piece is to be shaped into a square and the other piece into a circle. Where should the wire be cut to maximize the total area enclosed by the square and circle?



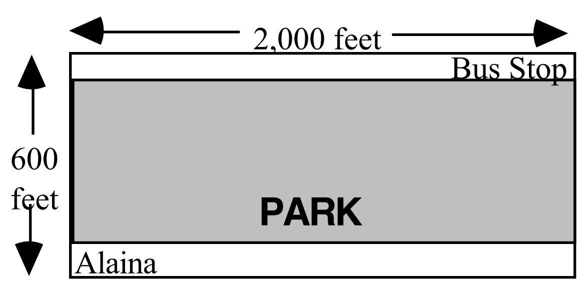
Let x be the point where the cut is made. Assume the square is on the left and the circle on the right. Complete the chart.

x	4	8	12	20	x
Area square					
Area circle					
Total area					

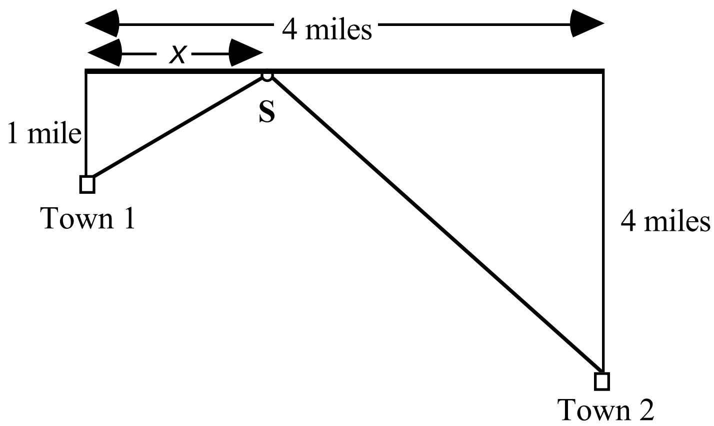
10. A designer of custom windows wishes to build a Norman Window with a total outside perimeter of 60 feet. How should the window be designed to maximize the area of the window. A Norman Window contains a rectangle bordered above by a semicircle.



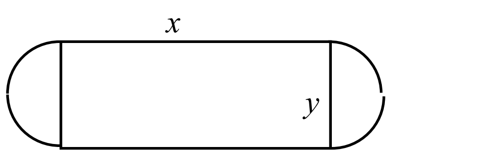
11. Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2,000 feet east and 600 feet north of her starting position. Alaina can walk along the edge of the park on the sidewalk at a speed of 6 feet/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec (dogs are walked here, so she must move with care). What path will get her to the bus stop the fastest.



12. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, **S**, that supplies water to them. The pumping station is to be at the river's edge with pipes extending straight to the two towns. The distances are shown in the figure below. Where should the pumping station be located to minimize the total length of pipe?



13. A physical fitness room consists of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a 200-meter running track, find the dimensions that will make the area of the rectangular region as large as possible.



14. Below is the graph of $y = 1 - x^2$. Find the point on this curve which is closest to the origin. (R*emember, you need a primary equation. What is it that you wish to minimize?*)