i) Find $f(a+h)$ and $f(a)$.

ii) Calculate $\frac{f(a+h)-f(a)}{h}$ and simplify if possible when $h \neq 0$.

iii) Find the limit of the result of part ii).

Example: Find the slope of the tangent line to $y = x^2$ at the point $a = 5$.

Using $m = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$:

i) $f(5) = 5^2 = 25$.

ii) $\frac{f(x)-f(5)}{x-5} = \frac{x^2-25}{x-5} = \frac{(x-5)(x+5)}{(x-5)} = x+5$, for $x \neq 5$.

iii) $m = \lim\limits_{x \to 5} \frac{f(x)-f(5)}{x-5} = \lim\limits_{x \to 5}(x+5) = 10$.

Using $m = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ we have:

i) $f(5+h) = (5+h)^2 = 25 + 10h + h^2$ and $f(5) = 25$.

ii) $\frac{f(5+h)-f(5)}{h} = \frac{(25+10h+h^2)-25}{h} = \frac{10h+h^2}{h} = \frac{h(10+h)}{h} = 10+h$, for $h \neq 0$.

iii) $m = \lim\limits_{h \to 0} \frac{f(5+h)-f(5)}{h} = \lim\limits_{h \to 0}(10+h) = 10$.

Notice that the direct substitution $x = a$ (or $h = 0$) in these limits always gives $\frac{0}{0}$.

1) Find the slope of the tangent line to $f(x) = x^2 + 2x$ at $x = 3$.

<First find f(3) and simplify $\frac{f(x)-f(3)}{x-3}$.>

i) At $x = 3$, $f(3) = 3^2 + 2(3) = 15$.

ii) $\frac{f(x)-f(a)}{x-a} = \frac{x^2+2x-15}{x-3} = \frac{(x-3)(x+5)}{x-3}$
$= x + 5$ for $x \neq 3$.

iii) Thus, $\lim\limits_{x \to 3} \frac{f(x)-f(3)}{x-3} = \lim\limits_{x \to 3}(x+5) = 8$.

The slope is 8.

2) Find $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ where $f(x) = \frac{1}{x-1}$ and $a = 2$.

<Construct the fraction $\frac{f(a+h)-f(a)}{h}$ *and simplify it. Then evaluate the limit.>*

i) $f(2+h) = \frac{1}{2+h-1} = \frac{1}{h+1}$

and $f(2) = \frac{1}{2-1} = \frac{1}{1}$.

2) *(Continued)*

ii) $\dfrac{f(2+h)-f(2)}{h}$

$$= \frac{\frac{1}{h+1}-\frac{1}{1}}{h} = \frac{\frac{1}{h+1}-\frac{h+1}{h+1}}{h}$$

$$= \frac{\frac{1-(h+1)}{h+1}}{h} = \frac{\frac{-h}{h+1}}{h} = \frac{-1}{h+1}, \text{ for } h \neq 0.$$

iii) $\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0} \dfrac{-1}{h+1} = -1.$

3) For $G(x) = 4x + 3$

a) Find the slope of the tangent line to G at $a = 2$.

i) $G(2) = 4(2) + 3 = 11.$

ii) $\dfrac{G(x)-G(2)}{x-2} = \dfrac{(4x+3)-11}{x-2}$

$$= \frac{4x-8}{x-2} = \frac{4(x-2)}{x-2}$$

$= 4$, for $x \neq 2$.

iii) $\lim\limits_{x\to 2} \dfrac{G(x)-G(2)}{x-2} = \lim\limits_{x\to 2} 4 = 4.$

b) What is the slope of the tangent line at any point a?

$G(x) = 4x + 3$ is a linear function, so the tangent line to G at $(a, G(a))$ will be the line itself, which has slope 4.

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B. $\dfrac{f(x)-f(a)}{x-a}$ is the **average rate of change of $y = f(x)$ on the interval $[a, x]$**. Thus

$\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ is the **instantaneous rate of change of $y = f(x)$ at $x = a$.**

In particular, if $y = f(x)$ is the position of an object at time x along an axis,

$\dfrac{f(x)-f(a)}{x-a}$ is the **average velocity over the time interval from a to x** and

$\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ is the **instantaneous velocity at $x = a$.**

4) Find the instantaneous velocity at time 3 seconds if a particle's position at time t is $f(t) = t^2 + 2t$ feet. See question 1.

8 ft/s. This question is asking for the same information as question 1. Here the rate of change is interpreted as velocity instead of slope.

5) a) A particle is moving along a scale in such a way that at time t, its position is $f(t) = 6t^2 - 4t + 1$ feet. Find the average velocity over each of these intervals of time:

<*Compute* $\frac{f(x) - f(a)}{x - a}$ *for each.*>

[1, 4]

$$\frac{f(4) - f(1)}{4 - 1} = \frac{81 - 3}{3} = 26 \text{ ft/s}$$

[1, 2]

$$\frac{f(2) - f(1)}{2 - 1} = \frac{17 - 3}{1} = 14 \text{ ft/s}$$

[1, 1.2]

$$\frac{f(1.2) - f(1)}{1.2 - 1} = \frac{4.84 - 3}{0.2} = 9.2 \text{ ft/s}$$

[1, 1.01]

$$\frac{f(1.01) - f(1)}{1.01 - 1} = \frac{3.0806 - 3}{0.01} = 8.06 \text{ ft/s}$$

b) Find the instantaneous velocity of the particle at $t = 1$ second.

<*Use the results from part a) to evaluate the limit as t approaches* 1.>

From part (a) it appears as though the answer is 8 ft/s. Let's find out:

$$\begin{aligned}\frac{f(t) - f(1)}{t - 1} &= \frac{6t^2 - 4t + 1 - 3}{t - 1} \\ &= \frac{6t^2 - 4t - 2}{t - 1} \\ &= \frac{(6t + 2)(t - 1)}{t - 1} \\ &= 6t + 2 \text{ for } t \neq 1.\end{aligned}$$

Thus,

$$\lim_{t \to 1} \frac{f(t) - f(1)}{t - 1} = \lim_{t \to 1} (6t + 2) = 8 \text{ ft/s}.$$

6) Fred is driving along a freeway. At 3:00 he passes mile marker 120 and at 5:00 he passes mile marker 250.

a) What was his average velocity during that time interval?

65 mi/hr.

b) When Fred passed mile marker 162, he saw that his speedometer read 54 mi/hr. What does 54 represent?

His instantaneous velocity at that time.

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C. The **derivative of $y = f(x)$ at a point a** is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

An equivalent way to write this is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If the function is given algebraically, such as $f(x) = x^2 + 2x + 1$, then $f'(a)$ is calculated in the same manner as in part A of this section. If the function is given in table form, then the derivative is estimated by first calculating the difference quotient $\frac{f(a+h) - f(a)}{h}$.

Example: Estimate the derivative at $a = 4$ for the function given at the right.

x	$f(x)$
0	1
2	5
4	8
6	11
8	13
10	16

First compute a table of difference quotients:

h	$a+h$	$f(a+h)$	$f(a+h) - f(a)$	$\frac{f(a+h) - f(a)}{h}$
−4	0	1	−7	$\frac{-7}{-4} = 1.75$
−2	2	5	−3	$\frac{-3}{-2} = 1.50$
2	6	11	3	$\frac{3}{2} = 1.50$
4	8	13	5	$\frac{5}{4} = 1.25$
6	10	16	8	$\frac{8}{6} = 1.33$

The table suggests that $f'(4)$ is somewhere between 1.25 and 1.75, and probably near 1.50 (since smaller values of h give difference quotients of 1.50). We estimate that $f'(4) \approx 1.50$.

7) True or False:

$$f'(a) = \lim_{h \to 0} \frac{f(h) - f(a)}{h}$$

False

8) Find $f'(3)$ for $f(x) = x^2 + 10x$.

<*Use the three steps to find $f'(x)$.*>

i) $f(3+h) = (3+h)^2 + 10(3+h)$
$= 9 + 6h + h^2 + 30 + 10h$
$= 39 + 16h + h^2$.
$f(3) = 3^2 + 10(3) = 39$.

ii) $\frac{f(3+h) - f(3)}{h} = \frac{(39 + 16h + h^2) - 39}{h}$
$= \frac{16h + h^2}{h} = \frac{h(16+h)}{h} = 16 + h$, for $h \neq 0$.

iii) Finally, $f'(3) = \lim_{h \to 0}(16 + h) = 16$.

9) Find $f'(a)$ for $f(x) = 2x - x^2$.

i) $f(a+h) = 2(a+h) - (a+h)^2$
$= 2a + 2h - a^2 - 2ah - h^2.$

$f(a) = 2a - a^2.$

ii) $\dfrac{f(a+h) - f(a)}{h}$

$= \dfrac{2a + 2h - a^2 - 2ah - h^2 - (2a - a^2)}{h}$

$= \dfrac{2h - 2ah - h^2}{h} = \dfrac{h(2 - 2a - h)}{h}$

$= 2 - 2a - h$ for $h \neq 0$.

iii) $f'(a) = \lim_{h\to 0}(2 - 2a - h)$
$= 2 - 2a - 0 = 2 - 2a.$

10) The population $p(t)$ of the village of Rosebush is given by the table below. Estimate the derivative for 1999.

t	*P*(*t*)
1996	841
1997	832
1998	821
1999	810
2000	801
2001	793

h	$a+h$	$f(a+h)$	$f(a+h) - f(a)$	$\frac{f(a+h)-f(a)}{h}$
−3	1996	841	31	−10.38
−2	1997	832	22	−11.00
−1	1998	821	11	−11.00
1	2000	801	−9	−9.00
2	2001	793	−17	−8.50

The difference quotients range from −8.50 to −11.00; we estimate $P'(1999) \approx -10.00$.

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D. The slope of the tangent line (measured by $f'(a)$) is the same as the **instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$.** The derivative measures how fast $f(x)$ is changing per unit change of x. Thus the derivative has many applications, such as velocity, metabolic rates, etc.

If $f'(a)$ exists, the tangent line to the graph of f at a is the line through $(a, f(a))$ with slope $f'(a)$. The tangent line has equation

$$y - f(a) = f'(a)(x - a).$$

If $f'(a)$ does not exist, then the tangent line might not exist, might be a vertical line, or might not be unique.

11) The slope of the tangent line to the graph of f at the point a is ______.

$f'(a)$, provided it exists.

12) True or False:
$f'(x)$ measures the average rate of change of $y = f(x)$ with respect to x.

False. $f'(x)$ measures instantaneous rate of change.

13) Given that $f'(x) = 3x^2 + 2$ for $f(x) = x^3 + 2x + 1$, find:

a) the slope of the tangent line to f at the point corresponding to $x = 2$.

The slope is $f'(2) = 3(2)^2 + 2 = 14$.

b) the equation of the tangent line to f at the point corresponding to $x = 2$.

<Use part a) to find the slope. Also find the y-coordinate corresponding to $x = 2$.>

The point of tangency has x-coordinate 2 and y-coordinate

$$f(2) = 2^3 + 2(2) + 1 = 13.$$

Hence the tangent line has equation

$$y - 13 = 14(x - 2) \text{ or } y = 14x - 15.$$

c) the instantaneous rate of change of $f(x)$ with respect to x at $x = 3$.

$f'(3) = 3(3)^2 + 2 = 29$.

d) If $f(x)$ represents the distance in feet of a particle from the origin at time x, find the instantaneous velocity at time $x = 4$ seconds.

$f'(4) = 3(4)^2 + 2 = 50$ ft/s.

14) An arrow is shot straight up into the air and falls back to earth after 8 seconds. Its height above the archer after t seconds is $H(t)$.

a) What is the meaning of $H'(2)$?

<Know that $H'(t)$ is the instantaneous velocity of H at time t.>

$H'(2)$ is the instantaneous rate of change of the height of the arrow above the archer after 2 seconds. The rate of change of height (distance) is velocity, so $H'(2)$ is the instantaneous velocity of the arrow after 2 seconds.

b) What is $H'(9)$?

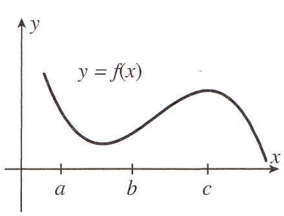
Since the arrow falls to earth after 8 seconds, it is not moving at time $t = 9$. Thus $H'(9) = 0$.

c) Is there a time a between 0 and 8 seconds for which $H'(a) = 0$?

Yes. $H'(a) = 0$ means the instantaneous velocity at time a is zero; that is, the arrow height does not change at time a.

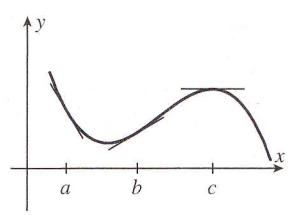
At its highest point, before the arrow starts returning to earth, there is an instant when $H'(a) = 0$. Since the entire flight takes 8 seconds, you might guess that at $a = 4$, $H'(4) = 0$.

15)



For the function f given above, which is largest: $f'(a), f'(b)$, or $f'(c)$?

<Use the fact that $f'(a)$ is the slope of the tangent line at $(a, f(a))$.>



$f'(b)$ is the largest. $f'(b)$ is positive while $f'(a) < 0$ and $f'(c) = 0$.

Section 3.2 The Derivative as a Function

For a function f, by letting the number a vary, the values of the derivative $f'(a)$ will vary. Thus, f' is itself a function. The function f' will turn out to provide a lot of useful information about the function f. This section includes several other common notations for the derivative, each of which traditionally has been used in certain applications. We will also see some cases where $f'(a)$ does not exist.

Concepts to Master

A. Differentiability; Other differentiation notation

B. Relationship between continuity and differentiability

C. Relationship between the graphs of f and f'

D. Determining where a function is not differentiable.

E. Second, third and higher derivative; Acceleration and jerk

Summary and Focus Questions

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A. By varying x, $f'(x)$ defines a function called **the derivative of f with respect to x:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The function **f is differentiable at x** means that $f'(x)$ exists.

f is differentiable on an interval if it is differentiable at every number in the interval.

All of these notations are used to refer to the derivative of f at x:

$$f'(x), \quad y', \quad \frac{df}{dx}, \quad \frac{dy}{dx}, \quad \frac{d}{dx}f(x), \quad Df(x), \quad D_x f(x)$$

1) True or False:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

True. This is the definition of $f'(x)$ using the symbol Δx instead of h.

2) Is $f(x) = \sqrt{x}$ differentiable at $x = 0$?

No. $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{h} - \sqrt{0}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}},$$

which does not exist.

3) True or False:

For $y = f(x)$ the notation $f'(x)$ and $\frac{dx}{dy}$ have the same meaning.

False. $f'(x) = \frac{dy}{dx}$.

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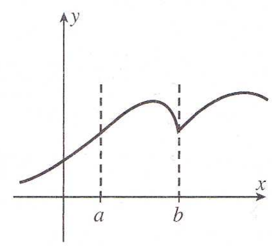
B. If f is differentiable at a point a, then f is also continuous at a. In terms of the graph for f, this means that if you can draw a tangent line to the graph at the point $(a, f(a))$, then the graph is unbroken at that point.

The converse is *false:* if a function f is continuous at a point b, it is not necessarily true that $f'(b)$ exists.

Example: The function graphed at the right is continuous at both a and b, differentiable at a, but not differentiable at b.



4) Can a function be continuous at $x = 3$ but not differentiable at $x = 3$?

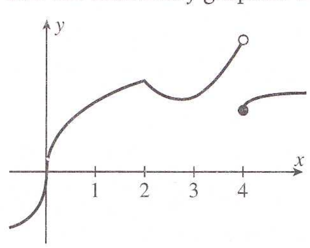
Yes. $f(x) = |x - 3|$ is an example. $f'(3)$ does not exist.

5) Can a function be differentiable at $x = 3$ and not continuous at $x = 3$?

No. This can never happen.

6) True or False:

For the function f graphed below:



<For continuity look for an unbroken curve. For differentiability, look for a smooth curve at which you could draw a non-vertical tangent line.>

a) f is continuous at 0.

True.

b) f is differentiable at 0.

False.

c) f is continuous at 2.

True.

d) f is differentiable at 2.

False.

e) f is continuous at 3.	True.
f) f is differentiable at 3.	True.
g) f is continuous at 4.	False.
h) f is differentiable at 4.	False.
i) f is differentiable on $[1, 3]$	False.
j) f is differentiable on $[2.5, 3.5]$	True.

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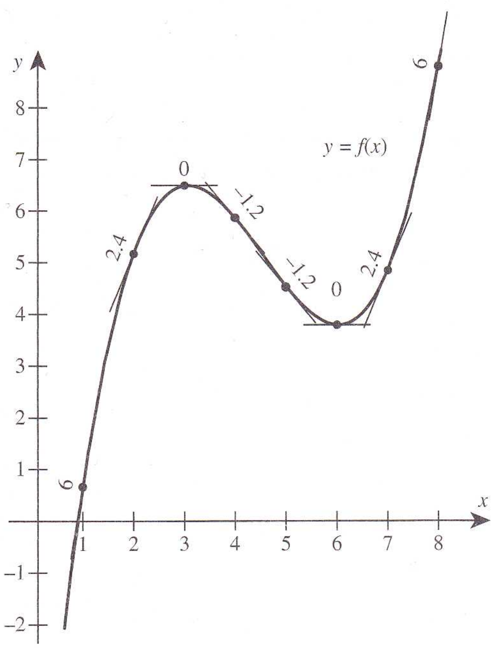
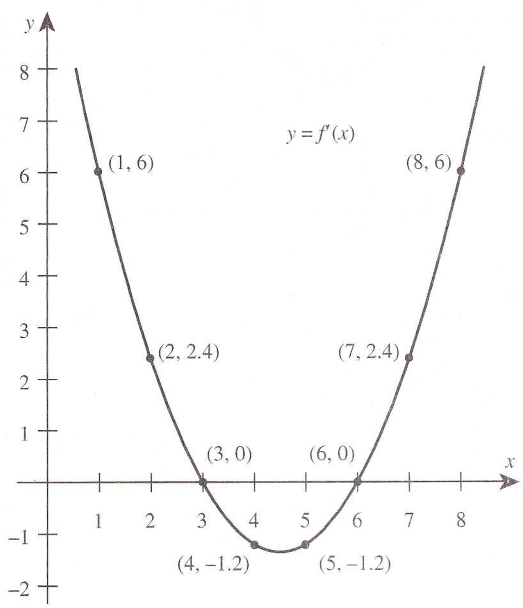
C. The graph of f' may be determined from the graph of f by remembering that $f'(x)$ is the slope of the tangent line at $(x, f(x))$:

If c is the slope of the tangent line at $(x, f(x))$, then (x, c) is on the graph of $y = f'(x)$.

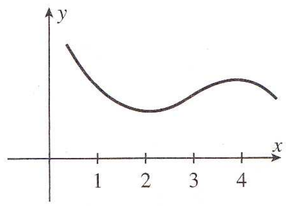
Conversely, if (x, c) is on the graph of f', then $f'(x) = c$ and c is the slope of the tangent line to f at $(x, f(x))$.

In the first graph below of $y = f(x)$, we have labeled some values of the slopes of tangent lines. The second graph is the graph of $y = f'(x)$.



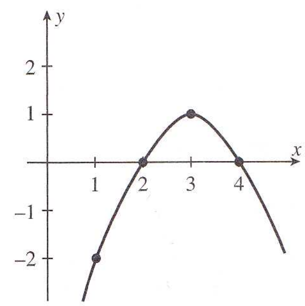


7) From the graph of f below, estimate $f'(1)$, $f'(2)$, $f'(3)$, and $f'(4)$, and then sketch a graph of $y = f'(x)$.

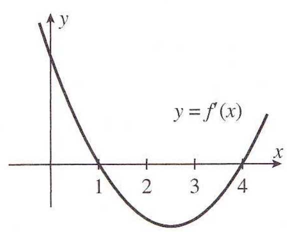


<Remember that $f'(x)$ is the slope of the tangent line.>

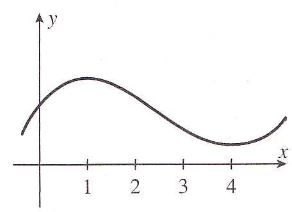
We estimate that $f'(1) = -2$, $f'(2) = 0$, $f'(3) = 1$, and $f'(4) = 0$. (Your estimates for $f'(1)$ and $f'(3)$ may be different.)For these estimates, the graph of f' is:



8) From the given graph of $y = f'(x)$, sketch a graph of $y = f(x)$.



One such graph of $y = f(x)$ is:

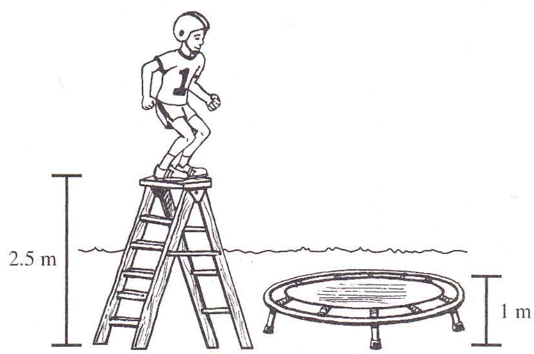


There are many other correct graphs, but all will have this shape.

9) Dan is on a ladder, 2.5 meters tall, above a trampoline which is 1 meter off the ground. He jumps onto the trampoline which propels him 3 meters into the air before he lands on the ground. Let $f(t)$ be his height after t seconds. Sketch a graph of $f(t)$ and $f'(t)$.

a) Sketch a graph of $y = f(t)$.

b) Sketch a graph of $y = f'(t)$.



<The important times are when Dan starts, when he reaches the trampoline, when the trampoline starts propelling him upward, when he reaches his highest point and when he lands on the ground.>

When Dan is on the top of the ladder, $t = 0$ and $f(0) = 2.5$.

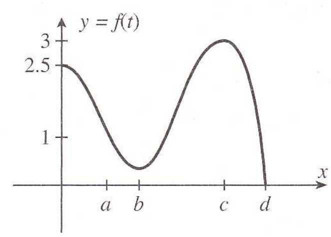
When Dan lands on the trampoline later, say at time $t = a, f(a) = 1$. For a very short time after that, he is sinking down below the trampoline. When the springs of the trampoline begin to propel Dan upward, the time is $t = b$, and $0 < f(b) < 1$.

Later $(t = c)$, when Dan reaches his apex (three meters high), $f(c) = 3$.

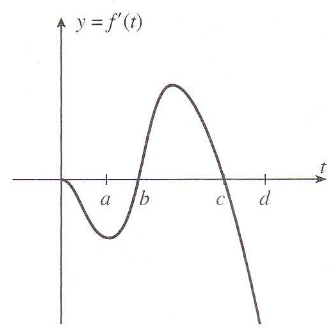
Finally, he lands on the ground at $t = d$, so $f(d) = 0$.

Between times 0 and b, Dan's velocity is negative because he is going down. Between times b and c, his velocity is positive because he is propelled upward. Between times c and d, he is headed back down so his velocity is negative.

a)



b)



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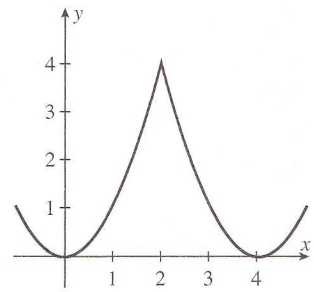
D. There are three common ways for a function to fail to be differentiable at a point.

i) The graph of the function has a corner or "kink" in it.

Example:

$$f(x) = \begin{cases} x^2 & x \le 2 \\ (x-4)^2 & x > 2 \end{cases}$$

f is not differentiable at $x = 2$.

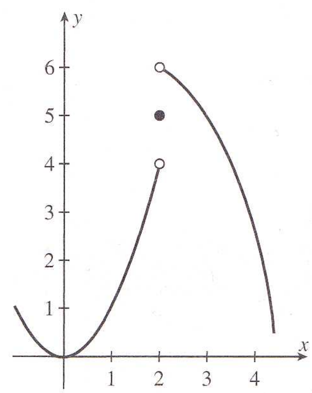


ii) The function is not continuous.

Example:

$$f(x) = \begin{cases} x^2 & x < 2 \\ 5 & x = 2 \\ 6-(x-2)^2 & x > 2 \end{cases}$$

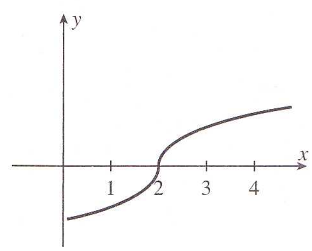
f is not differentiable at $x = 2$.



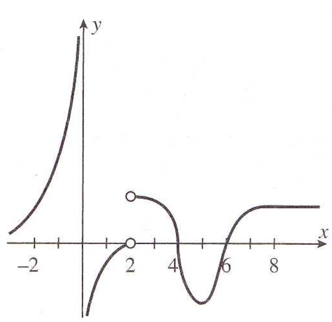
iii) The graph of the function has a tangent line, but it is a vertical line.

Example: $f(x) = \sqrt[3]{x-2}$

f is not differentiable at $x = 2$.



10) Where is $y = f(x)$ not differentiable?



It appears that f is not differentiable at $x = 0, 2, 4$.

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E. The **second derivative of $y = f(x)$,** denoted y'' or $f''(x)$, is the derivative of y'.
The **third derivative** is the derivative of the second derivative.
The ***n*th derivative of $y = f(x)$** is denoted $y^{(n)}$.
Acceleration is the rate of change of velocity. (The gas pedal in a car is called the "accelerator" because when you push your foot on it the car speeds up—the velocity changes—the acceleration is positive.)
The second derivative is a measure of acceleration.
The third derivative measures the rate of change of acceleration and is called the **jerk**, indicating a change (think of it as sudden change) in acceleration.
Each of these are notations for the *n*th derivative of y with respect to x:

$$y^{(n)},\ f^{(n)}(x),\ \frac{d^n y}{dx^n}$$

11) Find $f'(x)$ and $f''(x)$ for $f(x) = x^3 - 2x^2 + 3$.

<$f'(x)$ is found using the definition of the derivative. Once we have $f'(x)$ we use the definition again to find $f''(x)$.>

First find $f'(x)$.

i) $$\begin{aligned} f(x+h) &= (x+h)^3 - 2(x+h)^2 + 3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 2(x^2 + 2xh + h^2) + 3 \\ &= x^3 - 2x^2 + 3 + 3x^2h + 3xh^2 + h^3 - 4xh - 2h^2 \\ &= (x^3 - 2x^2 + 3) + h(3x^2 + 3xh + h^2 - 4x - 2h) \end{aligned}$$

ii) $$\begin{aligned} &\frac{f(x+h)-f(x)}{h} \\ &= \frac{[(x^3 - 2x^2 + 3) + h(3x^2 + 3xh + h^2 - 4x - 2h)] - (x^3 - 2x^2 + 3)}{h} \\ &= \frac{h(3x^2 + 3xh + h^2 - 4x - 2h)}{h} \\ &= 3x^2 + 3xh + h^2 - 4x - 2h, \text{ for } h \neq 0. \end{aligned}$$

iii) $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2 - 4x - 2h) = 3x^2 - 4x. \end{aligned}$$

Now find $f''(x)$ by the same process, using $f'(x)$ as our function.

i) $$\begin{aligned} f'(x+h) &= 3(x+h)^2 - 4(x+h) = 3(x^2 + 2xh + h^2) - 4x - 4h \\ &= 3x^2 + 6xh + 3h^2 - 4x - 4h \end{aligned}$$

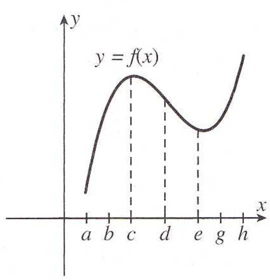
ii) $$\begin{aligned} &\frac{f'(x+h) - f'(x)}{h} = \frac{[(3x^2 + 6xh + 3h^2 - 4x - 4h)] - (3x^2 - 4x)}{h} \\ &= \frac{6xh + 3h^2 - 4h}{h} = \frac{h(6x + 3h - 4)}{h} = 6x + 3h - 4, \text{ for } h \neq 0. \end{aligned}$$

iii) $$\begin{aligned} f''(x) &= \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0}(6x + 3h - 4x) = 6x - 4. \end{aligned}$$

12) A particle moves along an axis in such a way that at time t the particle is $t^3 - 2t^2 + 3$ meters from the origin. Find the velocity and acceleration at time $t = 3$. Is the particle moving left or right? Is the particle speeding up or slowing down? Use question 11.

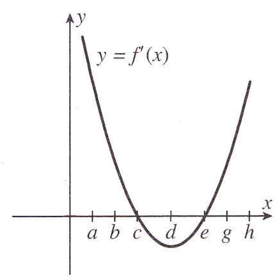
The position function is $f(t) = t^3 - 2t^2 + 3$. The velocity is $f'(t) = 3t^2 - 4t$ and at $t = 2$, $f'(2) = 3(2)^2 - 4(2) = 12 - 8 = 4$ m/sec. The acceleration is $f''(t) = 6t - 4$ and at $t = 2$, $f''(2) = 6(2) - 4 = 12 - 4 = 8$ m/sec^2. Since the velocity at $t = 2$ is positive, the particle is moving to the right. Since the acceleration at $t = 2$ is positive, the particle is increasing in speed.

13) Use your best judgment to determine the graph of $y = f''(x)$ given this graph of $y = f(x)$.



<$f'(x)$ is the slope of the tangent line to $y = f(x)$ and $f''(x)$ is the slope of the tangent line to $y = f'(x)$.>

From the slopes of tangents to $y = f(x)$ the graph of $y = f'(x)$ is:



From the slopes of tangents to this graph of $y = f'(x)$ the graph of $y = f''(x)$ is: