ZEROS OF POLYNOMIAL FUNCTIONS

Summary of Properties

1. The function given by $f(x) = a_n x^n + a_{n-1}x^{n-1} + x_{n-2}x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$ is called a **polynomial function** of x with **degree n**, where n is a nonnegative integer and $a_n,\ a_{n-1},\ a_{n-2},\ \dots,\ a_2,\ a_1,\ a_0$ are real numbers with a_n … 0.

2. The graphs of polynomial functions are **continuous** and have no sharp corners. The sign of the **leading coefficient** a_n determines the **end behavior** of the function. The degree n determines the number of complex zeros of the function. The number of real zeros of the function will be less than or equal to the number of complex zeros.

3. The **real zeros** of a polynomial function may be found by factoring (where possible) or by finding where the graph touches the x-axis. The number of times a zero occurs is called its **multiplicity**. If a function has a zero of odd multiplicity, the graph of the function crosses the x-axis at that x-value. However, if a function has a zero of even multiplicity, the graph of the function only touches the x-axis at that x-value.

4. The graphing calculator has a built-in function for finding a **zero** (or **root**) of a function. As an alternative method, you can graph y = 0 (the x-axis) as a second function and use the **intersection** function to find the zero. While this latter method is somewhat easier to use on some calculators, it may not work for finding zeros of even multiplicity.

Finding the Zeros of Polynomial Functions

Find the real zeros and state the multiplicity of each for the following polynomial functions:

Algebraic solution | Graphical solution

1. $f(x) = 4x^2 - 3x - 7$

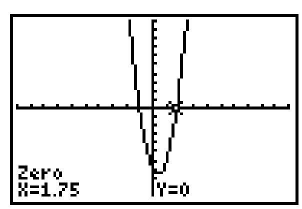
$4x^2 - 3x - 7 = 0$

$(4x - 7)(x + 1) = 0$

$4x - 7 = 0 \quad \text{or} \quad x + 1 = 0$

$x = \frac{7}{4} \quad \text{or} \quad x = -1$

Each zero has multiplicity one.



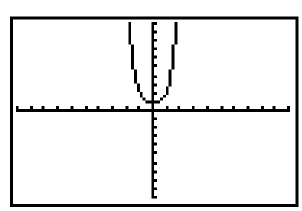
Repeat to find other zero

algebraic solution | graphical solution

2. $f(x) = x^4 + 1$

$x^4 + 1 = 0$

$x^4 = -1$ has no real solutions



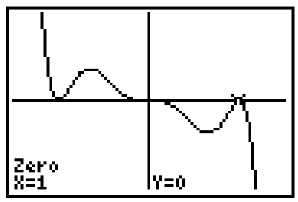
This function has no real zeros.

algebraic solution

3. $f(x) = -x^7 + 2x^5 - x^3$

$$-x^7 + 2x^5 - x^3 = 0$$
$$-x^3(x^4 - 2x^2 + 1) = 0$$
$$-x^3(x^2 - 1)^2 = 0$$
$$-x^3(x - 1)(x + 1)(x - 1)(x + 1) = 0$$
$$-x^3 = 0 \quad \text{or} \quad (x-1)^2 = 0 \quad \text{or} \quad (x+1)^2 = 0$$
$$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1$$

graphical solution



The zeros of the function are 0 (multiplicity 3), 1 (multiplicity 2), and –1 (multiplicity 1).

Writing Polynomial Functions with Specified Zeros

1. Write an equation of a polynomial function of degree 3 which has zeros of 0, 2, and –5.

   General solution: Any function of the form $f(x) = ax(x - 2)(x + 5)$ where a … 0 will have the required zeros.
   Specific solutions: $f(x) = x(x - 2)(x + 5) = x^3 + 3x^2 - 10x$
   $g(x) = -3x(x - 2)(x + 5) = -3x^3 - 9x^2 + 30x$

2. Write an equation of a polynomial function of degree 7 which has zeros of 0 (multiplicity 2), 2 (multiplicity 3), and –5 (multiplicity 2).

   General solution: Any function of the form $f(x) = ax^2(x - 2)^3(x + 5)^2$ where a … 0 will have the required zeros.

3. Write an equation of a polynomial function of degree 2 which has zero 4 (multiplicity 2) and opens downward.

   A typical solution is $f(x) = -3(x - 4)^2$. The leading coefficient must be negative.

4. Write an equation of a polynomial function of degree 3 which has zeros of –2, 2, and 6 and which passes through the point (3, 4).

   Solution: $f(x) = a(x + 2)(x - 2)(x - 6)$ has the required zeros

$f(3) = a(3 + 2)(3 - 2)(3 - 6) = 4$ **Y** $-15a = 4$ **Y** $a = -\frac{4}{15}$

$f(x) = -\frac{4}{15}(x + 2)(x - 2)(x - 6)$ has the required zeros and passes through the specified point.

Exercises

A. Algebraically find the exact real zeros and state the multiplicity of each.

1. $f(x) = 2x^2 + 9x - 5$

2. $f(x) = 9x^2 + 24x + 16$

3. $f(x) = 9x^2 + 4$

4. $f(x) = 9x^2 - 4$

5. $f(x) = 2x^2 - 4x + 1$

6. $f(x) = 8x^3 + 27$

7. $f(x) = 3x^5 + 5x^4 - x^3$

8. $f(x) = 32x^3 - 4$

9. $f(x) = 1 - x^4$

10. $f(x) = 2x^4 - 26x^2 + 72$

11. $f(x) = 4x^4 + 36$

12. $f(x) = x^3(2x + 1)^3(x^4 - 16)$

B. Graphically find the real zeros and state the multiplicity of each. Round answers to 4 decimal places.

13. $f(x) = x^2 + x - 1$

14. $f(x) = x^3 - 3x^2 + 2x - 4$

15. $f(x) = x^3 - 4x^2 + 2x + 1$

16. $f(x) = -x^3 + .5858x^2 + 1.8284x - 1.4142$

17. $f(x) = -x^4 + 8$

18. $f(x) = \frac{1}{6}x^4 - \frac{1}{3}x^2 + \frac{1}{6}$

19. $f(x) = 10000x^3 + 2x^2 - .001x - .000002$

C. Write an equation of a polynomial function which satisfies the given conditions. Answers are not unique.

20. Degree = 3; zeros are 0 (multiplicity 1) and 3 (multiplicity 2)

21. Degree = 4; zeros are 1, -6, $\frac{2}{3}$, and 4

22. Degree = 3; zeros are -5, 0, and 5; passes through (2, 21)

23. Degree = 4; zero = 2 (multiplicity 4); opens downward

D. Make a reasonable sketch of the graph of the most general polynomial function which satisfies the given conditions.

24. Degree = 3; has a zero of 3 with multiplicity 2; leading coefficient is positive

25. Degree = 4; has zeros of -2, 2, and 0 (multiplicity 2); leading coefficient is positive

26. Degree = 5; only real zero is -4 (multiplicity 1); leading coefficient is negative

27. Degree = 6; has zeros of 2 (multiplicity 3) and -4 (multiplicity 3); leading coefficient is negative

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