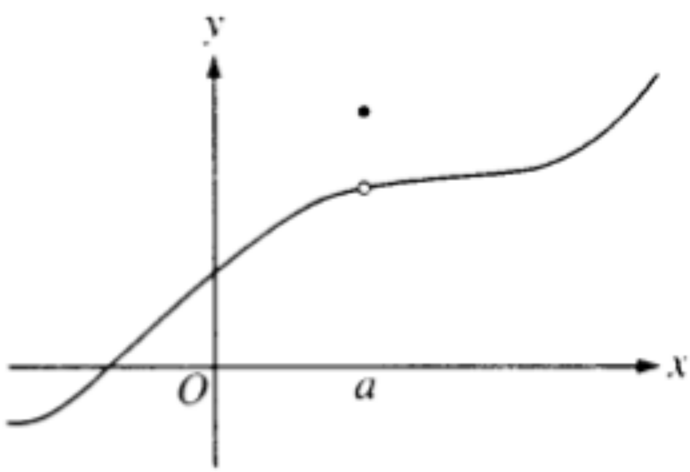
12. If $f(x)=\begin{cases} \ln x & \text{for } 0 < x \le 2 \\ x^2 \ln 2 & \text{for } 2 < x \le 4, \end{cases}$ then $\lim_{x \to 2} f(x)$ is

(A) $\ln 2$ (B) $\ln 8$ (C) $\ln 16$

(D) 4 (E) nonexistent



76. The graph of a function f is shown above. Which of the following statements about f is false?

(A) f is continuous at $x = a$.

(B) f has a relative maximum at $x = a$.

(C) $x = a$ is in the domain of f.

(D) $\lim_{x \to a^+} f(x)$ is equal to $\lim_{x \to a^-} f(x)$.

(E) $\lim_{x \to a} f(x)$ exists.

12. E $\lim_{x \to 2^-} f(x) = \ln 2 \neq 4 \ln 2 = \lim_{x \to 2^+} f(x)$.

Therefore the limit does not exist.

76. A From the graph it is clear that f is not

continuous at $x = a$. All others are true.

83. If $a \neq 0$, then $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x^4 - a^4}$ is

(A) $\dfrac{1}{a^2}$ (B) $\dfrac{1}{2a^2}$ (C) $\dfrac{1}{6a^2}$

(D) 0 (E) nonexistent

28. $\lim\limits_{x \to 1} \dfrac{\int_1^x e^{t^2}\,dt}{x^2 - 1}$ is

(A) 0 (B) 1 (C) $\dfrac{e}{2}$

(D) e (E) nonexistent

16. $\lim\limits_{h \to 0} \dfrac{e^h - 1}{2h}$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) 1

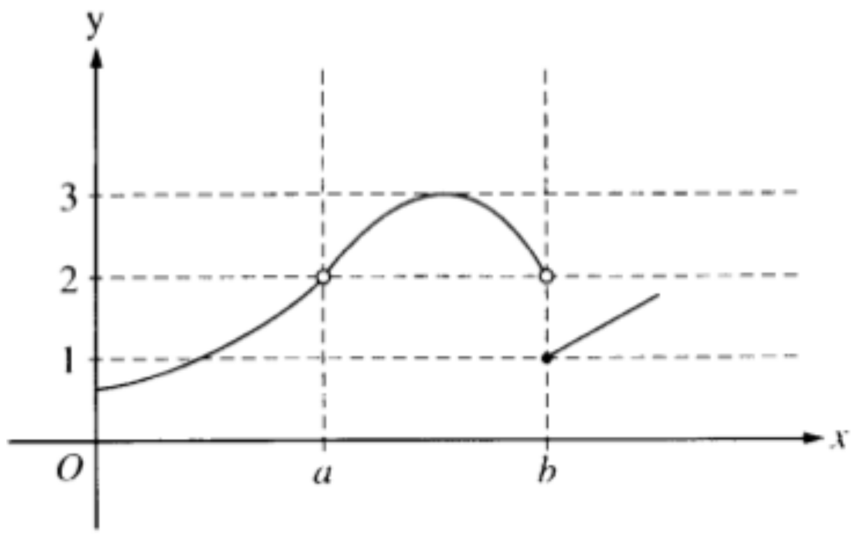
(D) e (E) nonexistent

83. B $\lim_{x\to a}\frac{x^2-a^2}{x^4-a^4}=\lim_{x\to a}\frac{x^2-a^2}{(x^2-a^2)(x^2+a^2)}=$

$$=\lim_{x\to a}\frac{1}{(x^2+a^2)}=\frac{1}{2a^2}$$

28. C Apply L'Hôpital's rule. $\lim_{x\to 1}\frac{\int_1^x e^{t^2}\,dt}{x^2-1}=\lim_{x\to 1}\frac{e^{x^2}}{2x}=\frac{e}{2}$

16. B $\lim_{h\to 0}\frac{e^h-1}{2h}=\frac{1}{2}$



15. The graph of the function f is shown in the figure above. Which of the following statements about f is true?

(A) $\lim_{x \to a} f(x) = \lim_{x \to b} f(x)$

(B) $\lim_{x \to a} f(x) = 2$

(C) $\lim_{x \to b} f(x) = 2$

(D) $\lim_{x \to b} f(x) = 1$

(E) $\lim_{x \to a} f(x)$ does not exist.

21. $\lim_{x \to 1} \dfrac{x}{\ln x}$ is

(A) 0 (B) $\dfrac{1}{e}$ (C) 1

(D) e (E) nonexistent

15. B Statement B is true because $\lim_{x\to a^-} f(x) = 2 = \lim_{x\to a^+} f(x)$.

Also, $\lim_{x\to b} f(x)$ does not exist because the left- and right-sided limits are equal, so neither (A), (C), nor (D) are true.

21. E $\lim_{x\to 1} \frac{x}{\ln x}$ is nonexistent since $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} x \neq 0$.

24. Let f and g be functions that are differentiable for all real numbers, with $g(x) \neq 0$ for $x \neq 0$. If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is

(A) 0

(B) $\frac{f'(x)}{g'(x)}$

(C) $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$

(D) $\frac{f'(x)g(x) - f(x)g'(x)}{\left(f(x)\right)^2}$

(E) nonexistent

42. $\lim_{x\to 0} (1+2x)^{\csc x} =$

(A) 0 (B) 1 (C) 2

(D) e (E) e^2

3. $\lim_{n\to\infty} \frac{3n^3 - 5n}{n^3 - 2n^2 + 1}$ is

(A) –5 (B) –2 (C) 1

(D) 3 (E) nonexistent

24. C This is L'Hôpital's Rule.

42. E

Suppose $\lim_{x\to 0} \ln\left((1+2x)^{\csc x}\right) = A$. The answer to the given question is e^A.

Use L'Hôpital's Rule: $\lim_{x\to 0} \ln\left((1+2x)^{\csc x}\right) = \lim_{x\to 0} \frac{\ln(1+2x)}{\sin x} = \lim_{x\to 0} \frac{2}{1+2x} \cdot \frac{1}{\cos x} = 2$.

3. D Divide each term by n^3. $\lim_{n\to\infty} \frac{3n^3 - 5n}{n^3 - 2n^2 + 1} = \lim_{n\to\infty} \frac{3 - \frac{5}{n^2}}{1 - \frac{2}{n} + \frac{1}{n^3}} = 3$

29. $\lim_{\theta \to 0} \frac{1-\cos\theta}{2\sin^2\theta}$ is

(A) 0 (B) $\frac{1}{8}$ (C) $\frac{1}{4}$

(D) 1 (E) nonexistent

20. The statement "$\lim_{x \to a} f(x) = L$" means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

(A) if $0 < |x-a| < \varepsilon$, then $|f(x)-L| < \delta$

(B) if $0 < |f(x)-L| < \varepsilon$, then $|x-a| < \delta$

(C) if $|f(x)-L| < \delta$, then $0 < |x-a| < \varepsilon$

(D) $0 < |x-a| < \delta$ and $|f(x)-L| < \varepsilon$

(E) if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$

29. C Use L'Hôpital's Rule

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{2\sin^2\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{4\sin\theta\cos\theta}=\lim_{\theta\to 0}\frac{1}{4\cos\theta}=\frac{1}{4}$$

A way to do this without L'Hôpital's rule is the following

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{2\sin^2\theta}=\lim_{\theta\to 0}\frac{1-\cos\theta}{2(1-\cos^2\theta)}=\lim_{\theta\to 0}\frac{1-\cos\theta}{2(1-\cos\theta)(1+\cos\theta)}=$$

$$=\lim_{\theta\to 0}\frac{1}{2(1+\cos\theta)}=\frac{1}{4}$$

20. E This is the definition of a limit.

35. If k is a positive integer, then $\lim_{x \to +\infty} \frac{x^k}{e^x}$ is

(A) 0 (B) 1 (C) e

(D) $k!$ (E) nonexistent

---

15. What is $\lim_{x \to 0} \frac{1 - e^{3x}}{\tan x}$

(A) –1
(B) –3
(C) 1
(D) 2
(E) The limit does not exist.

No Calculator

16. $\lim_{x \to 0} \frac{\ln(x + 1) - x}{x^2} =$

(A) 1

(B) 0

(C) $-\frac{1}{4}$

(D) $\frac{1}{2}$

(E) $-\frac{1}{2}$

35. A

Quick solution: For large x the exponential function dominates any polynomial, so $\lim_{x\to+\infty}\frac{x^k}{e^x}=0$.

15. B p. 6

The given limit has the indeterminate form $\frac{0}{0}$. Hence we can use L'Hôpital's Rule: $\lim_{x\to 0}\frac{1-e^{3x}}{\tan x}=\lim_{x\to 0}\frac{-3e^{3x}}{\sec^2 x}=-3$

16. E p. 50

The limit has the indeterminate form $\frac{0}{0}$.

By L'Hôpital's Rule: $\lim_{x\to 0}\frac{\ln(x+1)-x}{x^2}=\lim_{x\to 0}\frac{\frac{1}{x+1}-1}{2x}$

$$=\lim_{x\to 0}\frac{-\frac{1}{(x+1)^2}}{2}=-\frac{1}{2}$$

Calculator Active

5. The graph of $y = \frac{\sin x}{x}$ has

I. a vertical asymptote at $x = 0$

II. a horizontal asymptote at $y = 0$

III. an infinite number of zeros

(A) I only (B) II only (C) III only (D) I and III only (E) II and III only

No Calculator

2. $\lim_{x \to 3} \frac{(3-x)^2}{(x-3)}$ is

(A) 0

(B) -2

(C) 1

(D) -1

(E) nonexistent

5. E p. 55

Consider the function $y = \frac{\sin x}{x}$.

I. It has a removable discontinuity at $x = 0$. False

II. $\lim_{x \to \infty} \frac{\sin x}{x} = 0$. True

III. It has zeros at $x = \pm n\pi$, where n is an integer. True

2. Solution I: $\lim_{x \to 3} \frac{(3-x)^2}{x-3} = \lim_{x \to 3} -(3-x) = 0.$

Solution II: Since the given limit problem is of the form 0/0, L'Hôpital's Rule may be used.

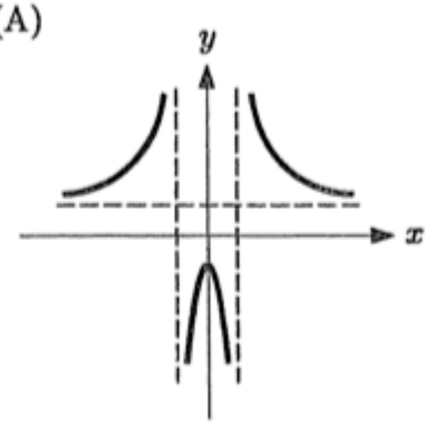
$$\lim_{x \to 3} \frac{(3-x)^2}{x-3} = \lim_{x \to 3} \frac{2(3-x)(-1)}{1} = 0$$

The correct choice is (A).

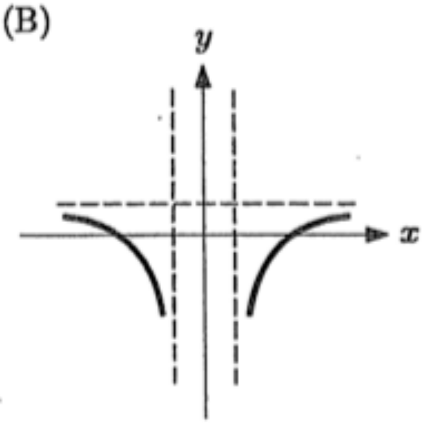
No Calculator

7. Which of the following graphs best resembles $y = \dfrac{|x|+1}{|x|-1}$?

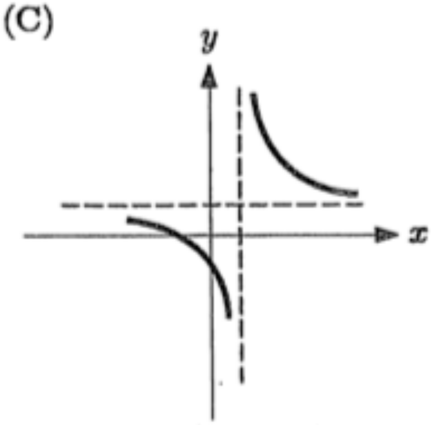
(A)



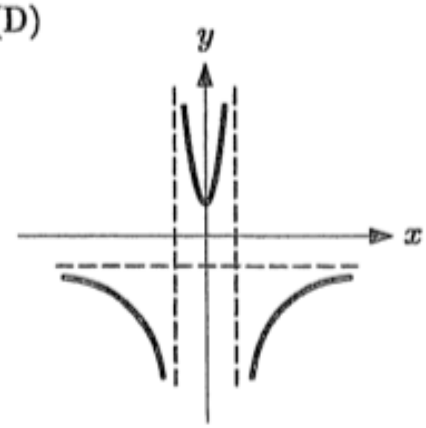
(B)



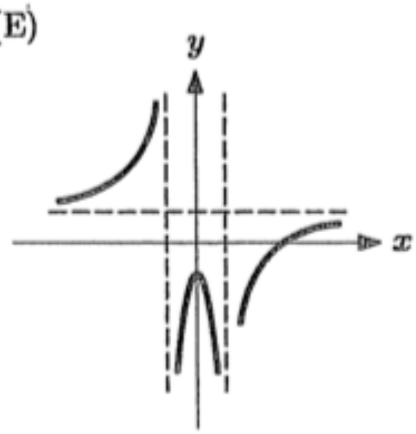
(C)



(D)



(E)



7. First, it should observed that the graph of $y = \frac{|x|+1}{|x|-1}$ has two vertical asymptotes, $x = -1$ and $x = 1$ (As $x \to 1$, $y \to \infty$). This rules out choices (B) and (C). Next, to find the y–intercept, set $x = 0$ and solve for y. If $x = 0$, $y = -1$, and the graph must have a y–intercept at $y = -1$. This rules out choice (D). Next, to find the x–intercept, set $y = 0$ and solve for x. If $y = 0$, then $|x| + 1 = 0$ or $|x| = -1$ which is impossible. Therefore, the graph has no x–intercepts — which rules out choice (E). The student may find other approaches to determine the graph of $y = \frac{|x|+1}{|x|-1}$.

The correct choice is (A).

Calculator Active

34. Which of the following functions grow faster than e^x as $x \to \infty$?

(A) x^4

(B) $\ln x$

(C) e^{-x}

(D) 3^x

(E) $\frac{1}{2}e^x$

34. A function $f(x)$ grows faster than $g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. In our problem, a function, say $f(x)$, will grow faster than e^x as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{e^x} = \infty$.

Let's examine the choices:

(A) $\lim_{x\to\infty} \frac{x^4}{e^x} = 0$ (Graph $\frac{x^4}{e^x}$ and see that as $x \to \infty$, $\frac{x^4}{e^x} \to 0$).

Note: Exponential functions like 2^x and e^x grow faster than polynomial functions as $x \to \infty$. For that matter, e^x grows faster than any power of x, say $x^{100{,}000{,}000}$ as $x \to \infty$.

(B) $\lim_{x\to\infty} \frac{\ln x}{e^x} = 0$ (Graph $\frac{\ln x}{e^x}$ and see that as $x \to \infty$, $\frac{\ln x}{e^x} \to 0$)

Note: $\ln x$ grows slower than any polynomial as $x \to \infty$, and (obviously) slower than e^x as $x \to \infty$.

(C) $\lim_{x\to\infty} \frac{e^{-x}}{e^x} = \lim_{x\to\infty} \frac{1}{e^{2x}} = 0$ (Note: $\frac{e^{-x}}{e^x} = \frac{\frac{1}{e^x}}{e^x} = \frac{1}{e^{2x}}$)

(D) $\lim_{x\to\infty} \frac{3^x}{e^x} = \lim_{x\to\infty} \left(\frac{3}{e}\right)^x = \infty$. So 3^x grows faster than e^x as $x \to \infty$.

Keep in mind, that if $a > b > 0$, then a^x always grows faster than b^x as $x \to \infty$. Since $a > b > 0$, so $\left(\frac{a}{b}\right) > 1$, and $\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty$.

(E) $\lim_{x\to\infty} \frac{\frac{1}{2}e^x}{e^x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$.

We say that f and g grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$, ($L$ is finite and not zero).

In this problem, the limit is $\frac{1}{2}$, so $f(x)$ grows at the same rate as $g(x)$, but $f(x)$ does not grow faster than $g(x)$ as $x \to \infty$.

Note: This problem could have been solved by graphing the functions in the window $[0, 10]$ by $[0, 6500]$. You will see that 3^x is the fastest growing function. Be careful: x^4 appears greater than e^x, but the functions cross at $x \approx 8.6$. Remember, every exponential function grows faster than any polynomial function.

The correct choice is (D).

14. A function $f(x)$ has a vertical asymptote at $x = 2$. The derivative of $f(x)$ is positive for all $x \neq 2$. Which statement is true?

I. $\lim_{x \to 2} f(x) = +\infty$

II. $\lim_{x \to 2^+} f(x) = +\infty$

III. $\lim_{x \to 2^-} f(x) = +\infty$

(A) I only

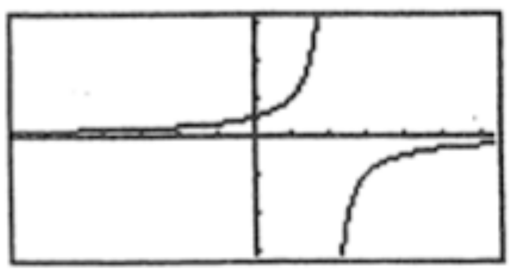
(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

14. Since the function has a derivative for all $x \neq 2$, it is continuous everywhere else. The derivative is positive indicating that the function is always increasing. The function must increase to infinity on the left side of the asymptote and then increase *from* $-\infty$ on the right side. In other words, this is an odd vertical asymptote. Thus the limit in II should equal $-\infty$ and the limit in I does not exist since the two one-sided limits are different. Only III is true. An example of such a function is $f(x) = \frac{-1}{x-2}$; its graph is shown below:



The correct choice is (C).