Exercises

1. What are the horizontal asymptotes of the function in Figure 6?

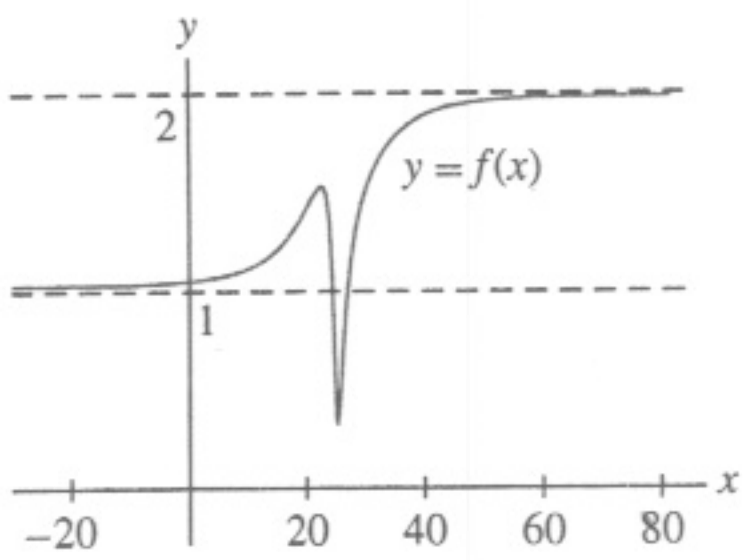


FIGURE 6

2. Sketch the graph of a function $f(x)$ that has both $y = -1$ and $y = 5$ as horizontal asymptotes.

3. Sketch the graph of a function $f(x)$ with a single horizontal asymptote $y = 3$.

4. Sketch the graphs of two functions $f(x)$ and $g(x)$ that have both $y = -2$ and $y = 4$ as horizontal asymptotes but $\lim_{x\to\infty} f(x) \neq \lim_{x\to\infty} g(x)$.

4. GU Investigate the asymptotic behavior of $f(x) = \dfrac{x^3}{x^3 + x}$ numerically and graphically:

(a) Make a table of values of $f(x)$ for $x = \pm 50, \pm 100, \pm 500, \pm 1000$.

(b) Plot the graph of $f(x)$.

(c) What are the horizontal asymptotes of $f(x)$?

5. GU Investigate $\lim_{x\to\pm\infty} \dfrac{12x + 1}{\sqrt{4x^2 + 9}}$ numerically and graphically:

(a) Make a table of values of $f(x) = \dfrac{12x + 1}{\sqrt{4x^2 + 9}}$ for $x = \pm 100$, [illegible], $\pm 1000, \pm 10{,}000$.

(b) Plot the graph of $f(x)$.

(c) What are the horizontal asymptotes of $f(x)$?

In Exercises 7–16, evaluate the limit.

$\lim_{x\to\infty} \dfrac{x}{x + 9}$

8. $\lim_{x\to\infty} \dfrac{3x^2 + 20x}{4x^2 + 9}$

$\lim_{x\to\infty} \dfrac{3x^2 + 20x}{2x^4 + 3x^3 - 29}$

10. $\lim_{x\to\infty} \dfrac{4}{x + 5}$

$\lim_{x\to\infty} \dfrac{7x - 9}{4x + 3}$

12. $\lim_{x\to\infty} \dfrac{9x^2 - 2}{6 - 29x}$

$\lim_{x\to-\infty} \dfrac{7x^2 - 9}{4x + 3}$

14. $\lim_{x\to-\infty} \dfrac{5x - 9}{4x^3 + 2x + 7}$

$\lim_{x\to-\infty} \dfrac{3x^3 - 10}{x + 4}$

16. $\lim_{x\to-\infty} \dfrac{2x^5 + 3x^4 - 31x}{8x^4 - 31x^2 + 12}$

In Exercises 17–22, find the horizontal asymptotes.

17. $f(x) = \dfrac{2x^2 - 3x}{8x^2 + 8}$

18. $f(x) = \dfrac{8x^3 - x^2}{7 + 11x - 4x^4}$

19. $f(x) = \dfrac{\sqrt{36x^2 + 7}}{9x + 4}$

20. $f(x) = \dfrac{\sqrt{36x^4 + 7}}{9x^2 + 4}$

21. $f(t) = \dfrac{e^t}{1 + e^{-t}}$

22. $f(t) = \dfrac{t^{1/3}}{(64t^2 + 9)^{1/6}}$

In Exercises 23–30, evaluate the limit.

23. $\lim_{x\to\infty} \dfrac{\sqrt{9x^4 + 3x + 2}}{4x^3 + 1}$

24. $\lim_{x\to\infty} \dfrac{\sqrt{x^3 + 20x}}{10x - 2}$

25. $\lim_{x\to-\infty} \dfrac{8x^2 + 7x^{1/3}}{\sqrt{16x^4 + 6}}$

26. $\lim_{x\to-\infty} \dfrac{4x - 3}{\sqrt{25x^2 + 4x}}$

27. $\lim_{t\to\infty} \dfrac{t^{4/3} + t^{1/3}}{(4t^{2/3} + 1)^2}$

28. $\lim_{t\to\infty} \dfrac{t^{4/3} - 9t^{1/3}}{(8t^4 + 2)^{1/3}}$

29. $\lim_{x\to-\infty} \dfrac{|x| + x}{x + 1}$

30. $\lim_{t\to-\infty} \dfrac{4 + 6e^{2t}}{5 - 9e^{3t}}$

31. Determine $\lim_{x\to\infty} \tan^{-1} x$. Explain geometrically.

32. Show that $\lim_{x\to\infty} (\sqrt{x^2 + 1} - x) = 0$. *Hint:* Observe that

$$\sqrt{x^2 + 1} - x = \frac{1}{\sqrt{x^2 + 1} + x}$$

33. According to the **Michaelis–Menten equation** (Figure 7), when an enzyme is combined with a substrate of concentration s (in millimolars), the reaction rate (in micromolars/min) is

$$R(s) = \frac{As}{K + s} \qquad (A, K \text{ constants})$$

(a) Show, by computing $\lim_{s\to\infty} R(s)$, that A is the limiting reaction rate as the concentration s approaches ∞.

(b) Show that the reaction rate $R(s)$ attains one-half of the limiting value A when $s = K$.

(c) For a certain reaction, $K = 1.25$ mM and $A = 0.1$. For which concentration s is $R(s)$ equal to 75% of its limiting value?

Leonor Michaelis 1875–1949 Maud Menten 1879–1960

FIGURE 7 Canadian-born biochemist Maud Menten is best known for her fundamental work on enzyme kinetics with German scientist Leonor Michaelis. She was also an accomplished painter, clarinetist, mountain climber, and master of numerous languages.

34. Suppose that the average temperature of the earth is $T(t) = 283 + 3(1 - e^{-0.03t})$ kelvins, where t is the number of years since 2000.

(a) Calculate the long-term average $L = \lim_{t\to\infty} T(t)$.

(b) At what time is $T(t)$ within one-half a degree of its limiting value?

In Exercises 35–42, calculate the limit.

35. $\lim_{x\to\infty} (\sqrt{4x^4 + 9x} - 2x^2)$

36. $\lim_{x\to\infty} (\sqrt{9x^3 + x} - x^{3/2})$

37. $\lim_{x\to\infty} (2\sqrt{x} - \sqrt{x+2})$

38. $\lim_{x\to\infty} \left(\frac{1}{x} - \frac{1}{x+2}\right)$

39. $\lim_{x\to\infty} (\ln(3x+1) - \ln(2x+1))$

40. $\lim_{x\to\infty} \left(\ln(\sqrt{5x^2+2}) - \ln x\right)$

41. $\lim_{x\to\infty} \tan^{-1}\left(\frac{x^2+9}{9-x}\right)$

42. $\lim_{x\to\infty} \tan^{-1}\left(\frac{1+x}{1-x}\right)$

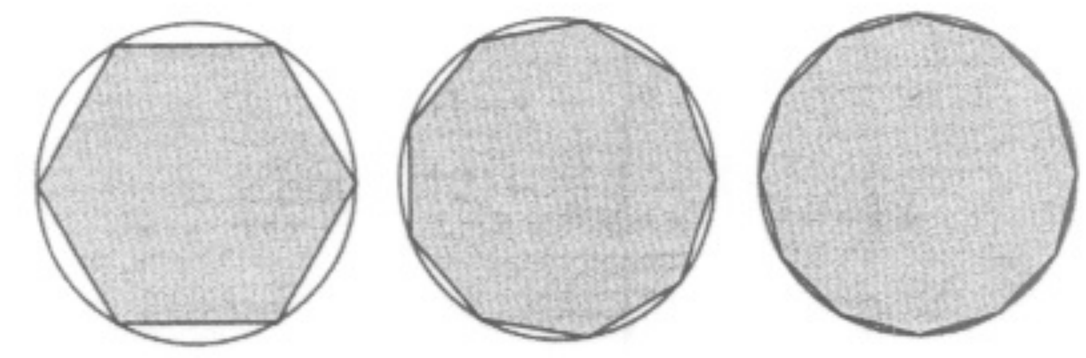
43. Let $P(n)$ be the perimeter of an n-gon inscribed in a unit circle (Figure 8).

(a) Explain, intuitively, why $P(n)$ approaches 2π as $n \to \infty$.

(b) Show that $P(n) = 2n \sin\left(\frac{\pi}{n}\right)$.

(c) Combine (a) and (b) to conclude that $\lim_{n\to\infty} \frac{n}{\pi} \sin\left(\frac{\pi}{n}\right) = 1$.

(d) Use this to give another argument that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.



$n = 6$ $n = 9$ $n = 12$

FIGURE 8

44. Physicists have observed that Einstein's theory of **special relativ ity** reduces to Newtonian mechanics in the limit as $c \to \infty$, where is the speed of light. This is illustrated by a stone tossed up verticall from ground level so that it returns to earth one second later. Usin Newton's Laws, we find that the stone's maximum height is $h = g/8$ meters ($g = 9.8$ m/s^2). According to special relativity, the stone's mas depends on its velocity divided by c, and the maximum height is

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g$$

Prove that $\lim_{c\to\infty} h(c) = g/8$.

Further Insights and Challenges

45. Every limit as $x \to \infty$ can be rewritten as a one-sided limit as $t \to 0+$, where $t = x^{-1}$. Setting $g(t) = f(t^{-1})$, we have

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0+} g(t)$$

Show that $\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0+} \frac{3 - t}{2 + 5t^2}$, and evaluate using the Quotient Law.

46. Rewrite the following as one-sided limits as in Exercise 45 and evaluate.

(a) $\lim_{x\to\infty} \frac{3 - 12x^3}{4x^3 + 3x + 1}$

(b) $\lim_{x\to\infty} e^{1/x}$

(c) $\lim_{x\to\infty} x \sin\frac{1}{x}$

(d) $\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right)$

47. Let $G(b) = \lim_{x\to\infty} (1 + b^x)^{1/x}$ for $b \geq 0$. Investigate $G(b)$ numer ically and graphically for $b = 0.2, 0.8, 2, 3, 5$ (and additional values i necessary). Then make a conjecture for the value of $G(b)$ as a functio of b. Draw a graph of $y = G(b)$. Does $G(b)$ appear to be continu ous? We will evaluate $G(b)$ using L'Hôpital's Rule in Section 4.5 (se Exercise 69 in Section 4.5).

2.8 Intermediate Value Theorem

The **Intermediate Value Theorem (IVT)** says, roughly speaking, that *a continuous func tion cannot skip values*. Consider a plane that takes off and climbs from 0 to 10,000 meter in 20 minutes. The plane must reach every altitude between 0 and 10,000 meters durin this 20-minute interval. Thus, at some moment, the plane's altitude must have been exactl 8371 meters. Of course, this assumes that the plane's motion is continuous, so its altitud cannot jump abruptly from, say, 8000 to 9000 meters.

To state this conclusion formally, let $A(t)$ be the plane's altitude at time t. The IVT asserts that for every altitude M between 0 and 10,000, there is a time t_0 between 0 an 20 such that $A(t_0) = M$. In other words, the graph of $A(t)$ must intersect the horizonta line $y = M$ [Figure 1(A)].

By contrast, a discontinuous function can skip values. The greatest integer functio $f(x) = [x]$ in Figure 1(B) satisfies $[1] = 1$ and $[2] = 2$, but it does not take on the valu 1.5 (or any other value between 1 and 2).

| A proof of the IVT is given in Appendix B.

x	$c-.01$	$c-.001$	$c+.001$	$c+.01$
$\frac{\sin x - \sin c}{x-c}$	.868511	.866275	.865775	.863511

Here $c = \frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx .866025$.

x	$c-.01$	$c-.001$	$c+.001$	$c+.01$
$\frac{\sin x - \sin c}{x-c}$	.504322	.500433	.499567	.495662

Here $c = \frac{\pi}{3}$ and $\cos c = \frac{1}{2}$.

x	$c-.01$	$c-.001$	$c+.001$	$c+.01$
$\frac{\sin x - \sin c}{x-c}$	.710631	.707460	.706753	.703559

Here $c = \frac{\pi}{4}$ and $\cos c = \frac{\sqrt{2}}{2} \approx 0.707107$.

x	$c-.01$	$c-.001$	$c+.001$	$c+.01$
$\frac{\sin x - \sin c}{x-c}$	.005000	.000500	−.000500	−.005000

Here $c = \frac{\pi}{2}$ and $\cos c = 0$.

(b) $\lim_{x\to c} \frac{\sin x - \sin c}{x-c} = \cos c$.

(c)

x	$c-.01$	$c-.001$	$c+.001$	$c+.01$
$\frac{\sin x - \sin c}{x-c}$	−.411593	−.415692	−.416601	−.420686

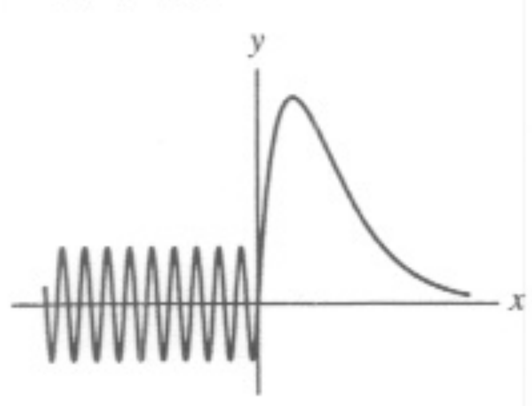
Here $c = 2$ and $\cos c = \cos 2 \approx -.416147$.

x	$c-.01$	$c-.001$	$c+.001$	$c+.01$
$\frac{\sin x - \sin c}{x-c}$	.863511	.865775	.866275	.868511

Here $c = -\frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx .866025$.

Section 2.7 Preliminary Questions

1. **(a)** Correct **(b)** Not correct **(c)** Not correct **(d)** Correct
2. **(a)** $\lim_{x\to\infty} x^3 = \infty$ **(b)** $\lim_{x\to-\infty} x^3 = -\infty$
(c) $\lim_{x\to-\infty} x^4 = \infty$
3.



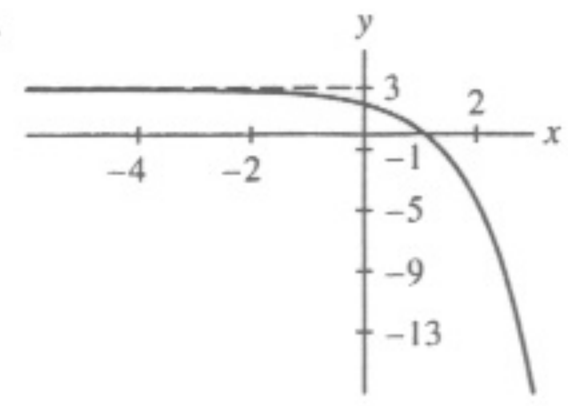
4. Negative **5.** Negative
6. As $x \to \infty$, $\frac{1}{x} \to 0$, so

$$\lim_{x\to\infty} \sin\frac{1}{x} = \sin 0 = 0.$$

On the other hand, $\frac{1}{x} \to \pm\infty$ as $x \to 0$, and as $\frac{1}{x} \to \pm\infty$, $\sin\frac{1}{x}$ oscillates infinitely often.

Section 2.7 Exercises

1. $y = 1$ and $y = 2$
3.



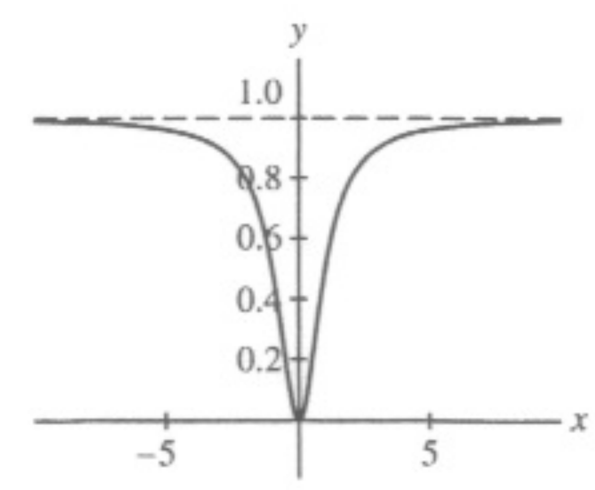
5. **(a)** From the table below, it appears that

$$\lim_{x\to\pm\infty} \frac{x^3}{x^3+x} = 1.$$

x	±50	±100	±500	±1000
$f(x)$	.999600	.999900	.999996	.999999

(b) From the graph below, it also appears that

$$\lim_{x\to\pm\infty} \frac{x^3}{x^3+x} = 1.$$



(c) The horizontal asymptote of $f(x)$ is $y = 1$.
7. 1 **9.** 0 **11.** $\frac{7}{4}$ **13.** $-\infty$ **15.** ∞ **17.** $y = \frac{1}{4}$ **19.** $y = \frac{2}{3}$ and $y = -\frac{2}{3}$ **21.** $y = 0$ **23.** 0 **25.** 2 **27.** $\frac{1}{16}$ **29.** 0
31. $\frac{\pi}{2}$; the graph of $y = \tan^{-1} x$ has a horizontal asymptote at $y = \frac{\pi}{2}$
33. **(a)** $\lim_{s\to\infty} R(s) = \lim_{s\to\infty} \frac{As}{K+s} = \lim_{s\to\infty} \frac{A}{1+\frac{K}{s}} = A$.

(b) $R(K) = \frac{AK}{K+K} = \frac{AK}{2K} = \frac{A}{2}$ half of the limiting value.

(c) 3.75 mM
35. 0 **37.** ∞ **39.** $\ln\frac{3}{2}$ **41.** $-\frac{\pi}{2}$
45. $\lim_{x\to\infty} \frac{3x^2-x}{2x^2+5} = \lim_{t\to 0+} \frac{3-t}{2+5t^2} = \frac{3}{2}$
47.
- $b = 0.2$:

x	5	10	50	100
$f(x)$	1.000064	1.000000	1.000000	1.000000

It appears that $G(0.2) = 1$.
- $b = 0.8$:

x	5	10	50	100
$f(x)$	1.058324	1.010251	1.000000	1.000000