17. $r = \frac{20}{25} = 0.8 < 1.\ S = \frac{25}{1-0.8} = 125$

18. $r = \frac{18}{15} > 1$. Thus, no sum exists.

19. $r = -\frac{2}{3}.\ \left|-\frac{2}{3}\right| < 1.\ S = -\frac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = -\frac{2}{5}$

20. $r = 0.22 < 1.\ a_1 = 4(0.22) = 0.88.$
$S = \frac{0.88}{1-0.22} = \frac{44}{39}$

21. Base Case: $4 = 4(1) = 2(1)(1 + 1) = 4$

22. Assume this is true for a natural number k. Thus:
$4 + 8 + ... + 4k = 2k(k + 1)$

23. Look at $n = k + 1$.
$4 + ... + 4k + 4(k + 1)$
$= 2k(k + 1) + 4(k + 1)$
$= 2k^2 + 2k + 4k + 1$
$= 2k^2 + 6k + 1$
$= (k + 1)(2k + 4)$
$= 2(k + 1)(k + 2)$

24a. Once the ball initially hits the ground, it rises (60%)(5) = 3 ft. Since it goes up and down, it is 2(3) feet. This is a_1. The ratio is 0.6. Thus, the series is:
$\sum_{k=1}^{\infty} 2(3)(0.6)^{k-1}$

b. This is an infinite geometric series with $r = 0.6$ and $a_1 = 6$. $S = \frac{6}{1-0.6} = \frac{1}{0.4} = 15$ ft.

STUDY GUIDE: REVIEW

1. arithmetic sequence; geometric sequence.
2. diverges; converges.
3. explicit formula; recursive formula
4. infinite sequence; finite sequence
5. iteration

9-1 INTRODUCTION TO SEQUENCES

6.

n	1	2	3	4	5
a_n	–8	–7	–6	–5	–4

7.

n	1	2	3	4	5
a_n	$\frac{1}{2}$	2	$\frac{9}{2}$	8	$\frac{25}{2}$

8.

n	1	2	3	4	5
a_n	1	$-\frac{3}{2}$	$\frac{9}{4}$	$-\frac{27}{8}$	$\frac{81}{16}$

9.

n	1	2	3	4	5
a_n	55	53	51	49	47

10.

n	1	2	3	4	5
a_n	200	40	8	$\frac{8}{5}$	$\frac{8}{25}$

11.

n	1	2	3	4	5
a_n	–3	10	–29	88	–263

12. Differences: –4, –4, –4, –4. So, the sequence is arithmetic, with $a_1 = -4$ and $d = -4$. So, the rule is $a_n = -4 - 4(n - 1) = -4 - 4n + 4 = -4n$.

13. Ratios: 4, 4, 4, 4. So, the sequence is geometric, with $a_1 = 5$ and $r = 4$. So, the rule is $a_n = 5(4)^{n-1}$

14. Differences: 5, 5, 5, 5. So, the sequence is arithmetic, with $a_1 = -24$ and $d = 5$. So, the rule is $a_n = -24 + 5(n - 1) = -24 + 5n - 5 = -29 + 5n$.

15. Ratios: $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}$. So, the sequence is geometric, with $a_1 = 27$, $r = \frac{2}{3}$. So, the rule is $a_n = 27\left(\frac{2}{3}\right)^{n-1}$

16. After the first bounce, the ball will rebound to (70%×3) = 2.1 ft. This is a_1 in the sequence, and $r = 0.7$. So, $a_n = 2.1(0.7)^{n-1}$.
$a_4 = 2.1(0.7)^3 = 0.72$ ft.
$a_9 = 2.1(0.7)^8 = 0.12$ ft.

9-2 SERIES AND SUMMATION NOTATION

17. $1^2(-1)^1 + 2^2(-1)^2 + 3^2(-1)^3 + 4^2(-1)^4$
$= -1 + 4 - 9 + 16 = 10$

18. $(0.5(1) + 4) + (0.5(2) + 4) + (0.5(3) + 4)$
$+ (0.5(4) + 4) + (0.5(5) + 4)$
$= 4.5 + 5 + 5.5 + 6 + 6.5 = 27.5$

19. $(-1)^{1-1}(2(1) - 1) + (-1)^{2-1}(2(2) - 1)$
$+ (-1)^{3-1}(2(3) - 1) + (-1)^{4-1}(2(4) - 1)$
$+ (-1)^{5-1}(2(5) - 1)$
$= 1 - 3 + 5 - 7 + 9 = 5$

20. $\frac{5(1)}{(1)^2} + \frac{5(2)}{(2)^2} + \frac{5(3)}{(3)^2} + \frac{5(4)}{(4)^2}$
$= 5 + \frac{10}{4} + \frac{15}{9} + \frac{20}{16} = 10\frac{5}{12}$

21. $8(-5) = -40$

22. $\frac{10(10+1)(2(10)+1)}{6} = \frac{10(11)(21)}{6} = \frac{2310}{6} = 385$

23. $\frac{12(12+1)}{2} = \frac{12(13)}{2} = 78$

24. We can model this with the formula \$1150(12n) where n is the number of years. So:
$a_2 = 1150(12(2)) = \$27,600$.
$a_{15} = 1150(12(15)) = \$207,000$.

9-3 ARITHMETIC SEQUENCES AND SERIES

25. $d = 19 - 23 = -4$.
$a_{11} = 23 - 4(11 - 1) = 23 - 40 = -17$

26. $d = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$
$a_{11} = \frac{1}{5} + \frac{2}{5}(11 - 1) = \frac{1}{5} + \frac{20}{5} = \frac{21}{5}$.

27. $d = -8.4 - (-9.2) = 0.8$
$a_{11} = -9.2 + 0.8(11 - 1) = -1.2$

28. $d = 5 - 1.5 = 3.5$.
$a_3 = 1.5 = a_1 + 3.5(3 - 1)$. So, $a_1 = -5.5$

$a_{11} = -5.5 + 3.5(11 - 1) = -5.5 + 35 = 29.5.$

29. $2d = 21 - 47 = -26$. So, $d = -13$.
$a_6 = 47 = a_1 - 13(6 - 1)$. So, $a_1 = 112$.
$a_{11} = 112 - 13(11 - 1) = 112 - 130 = -18.$

30. $4d = 13 - (-7) = 20$. So. $d = 5$.
$a_5 = -7 = a_1 + 5(5 - 1)$. So, $a_1 = -27$.
$a_{11} = -27 + 5(11 - 1) = -27 + 50 = 23.$

31. $a_1 = -1$. $a_{18} = -1 - 4(18 - 1) = -69$.
$S_{18} = 18\left(\frac{a_1 + a_{18}}{2}\right) = 18\frac{-1 - 69}{2} = -630$

32. $a_1 = \frac{1}{3}$. $a_{12} = \frac{1}{3} - \frac{1}{6}(12 - 1) = -\frac{3}{2}$.
$S_{12} = 12\left(\frac{a_1 + a_{12}}{2}\right) = 12\frac{\frac{1}{3} - \frac{3}{2}}{2} = -7$

33. $a_1 = -14 + 3(1) = -11$. $a_{15} = -14 + 3(15) = 31$.
$S_{15} = 15\left(\frac{a_1 + a_{15}}{2}\right) = 15\left(\frac{-11 + 31}{2}\right) = 150$

34. $a_1 = \frac{3}{2}(1) + 10 = 11\frac{1}{2}$. $a_{15} = \frac{3}{2}(15) + 10 = 32\frac{1}{2}$.
$S_{15} = 15\left(\frac{a_1 + a_{15}}{2}\right) = 15\left(\frac{11.5 + 32.5}{2}\right) = 330$

35. 50, 58, 66.... $a_1 = 50$. $d = 8$.
$a_{52} = 50 + 8(52 - 1) = 458$. Kelly would not have enough to be able to buy the bicycle.

9-4 GEOMETRIC SEQUENCES AND SERIES

36. $a_1 = 40$. $r = 0.1$. $a_8 = 40(0.1)^{8-1} = 0.000004$.

37. $a_1 = \frac{1}{18}$. $r = 3$. $a_8 = \frac{1}{18}(3)^{8-1} = 121.5$

38. $a_1 = -16$. $r = 0.5$. $a_8 = -16(0.5)^{8-1} = -\frac{1}{8}$.

39. $a_1 = -6$. $r = -2$. $a_8 = -6(-2)^{8-1} = 768$

40. $r = \frac{96}{24} = 4$.
$a_3 = 24 = a_1(4)^{3-1}$. So, $a_1 = 1.5$.
$a_9 = 1.5(4)^{9-1} = 98{,}304$.

41. $r = -2$.
$a_9 = \frac{2}{3}(-2)^{9-1} = \frac{512}{3}$

42. $-4 = -1r^{6-4}$. *So*, $r^2 = 4$. $r = \pm 2$.
If $r = 2$: $a_4 = -1 = a_1(2)^{4-1}$. $a_1 = -\frac{1}{8}$.
$a_9 = -\frac{1}{8}(2)^{9-1} = -32.$
If $r = -2$: $a_4 = -1 = a_1(-2)^4 - 1$. $a_1 = \frac{1}{8}$.
$a_9 = \frac{1}{8}(2)^{9-1} = 32.$
Thus, $a_9 = \pm 32$.

43. $500 = 4r^{6-3}$. *So*, $r^3 = 125$. $r = 5$.
$a_3 = 4 = a_1(5)^{3-1}$. *So*, $a_1 \frac{4}{25}$.
$a_9 = \frac{4}{25}(5)^{9-1} = 62{,}500.$

44. $\sqrt{ab} = \sqrt{10 \cdot 2.5} = \sqrt{25} = 5.$

45. $\sqrt{ab} = \sqrt{\frac{1}{2} \cdot 8} = \sqrt{4} = 2.$

46. $\sqrt{ab} = \sqrt{\frac{\sqrt{3}}{96}\frac{\sqrt{3}}{6}} = \sqrt{\frac{3}{576}} = \frac{\sqrt{3}}{24}.$

47. $\sqrt{ab} = \sqrt{\frac{5}{12}\frac{125}{108}} = \sqrt{\frac{625}{1296}} = \frac{25}{36}$

48. $r = 1/3$. $a_1 = 1$.
$S_5 = 1\left(\frac{1 - \left(\frac{1}{3}\right)^5}{1 - \frac{1}{3}}\right) = \frac{121}{81}$

49. $r = -10$. $a_1 = -\frac{4}{5}$
$S_6 = -\frac{4}{5}\left(\frac{1 - (-10)^6}{1 - (-10)}\right) = 72{,}727.2$

50. $r = 4$. $a_1 = 1$.
$S_8 = 1\left(\frac{1 - 4^8}{1 - 4}\right) = 21{,}845$

51. $r = 5$. $a_1 = -2$.
$S_7 = -2\left(\frac{1 - 5^7}{1 - 5}\right) = -39{,}062$

52. $r = -\frac{1}{2}$. $a_1 = 60$.
$S_6 = 60\left(\frac{1 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}}\right) = 39.375$

53. $r = -\frac{1}{2.}$ $a_1 = 18$.
$S_5 = 18\left(\frac{1 - \left(-\frac{1}{2}\right)^5}{1 - \left(-\frac{1}{2}\right)}\right) = 34.875$

54. $a_1 = 9000$. $r = 0.65$.
$a_5 = 9000(0.65)^5 = \$1044.26$

55a. $a_1 = 650 \times 12 = 7800$. $r = 1.06$
$a_5 = 7800(1.06)^{5-1} = \$9847.32$

b. $S_5 = 7800\left(\frac{1 - (1.06)^5}{1 - 1.06}\right) = \$43{,}969.32$

9-5 MATHEMATICAL INDUCTION AND INFINITE GEOMETRIC SERIES

56. $r = \frac{900}{-2700} = -\frac{1}{3}$. $\left|-\frac{1}{3}\right| < 1$. So, a sum exists.
$S = \frac{-2700}{1 - \left(-\frac{1}{3}\right)} = -\frac{2700}{\frac{4}{3}} = -2025.$

57. $r = \frac{-0.12}{-1.2} = 0.1 < 1$. So, a sum exists.
$S = \frac{-1.2}{1 - 0.1} = \frac{-1.2}{0.9} = -1.\overline{3}$

58. $r = \frac{-42}{-49} = \frac{6}{7} < 1$. So, a sum exists.
$S = \frac{-49}{1 - \frac{6}{7}} = \frac{-49}{\frac{1}{7}} = -343.$

59. $r = \frac{\frac{4}{5}}{4} = \frac{1}{5} < 1$. So, a sum exists.
$S = \frac{4}{1 - \frac{1}{5}} = \frac{4}{\frac{4}{5}} = \frac{20}{4} = 5$

60. $r = \frac{1}{3} < 1$. $a_1 = \frac{9}{3^1} = 3$.

$S = \frac{3}{1 - \frac{1}{3}} = \frac{3}{\frac{2}{3}} = \frac{9}{2} = 4.5$

61. $r = \frac{3}{5} < 1$. $a_1 = -7\left(\frac{3}{5}\right) = -\frac{21}{5}$.

$S = \frac{-\frac{21}{5}}{1 - \frac{3}{5}} = -\frac{\frac{21}{5}}{\frac{2}{5}} = -\frac{21}{2} = -10.5.$

62. $a_1 = (-1)^2\left(\frac{1}{8^1}\right) = \frac{1}{8}$. $r = -\frac{1}{8}$. $\left|-\frac{1}{8}\right| < 1$

$S = \frac{\frac{1}{8}}{1 - \left(-\frac{1}{8}\right)} = \frac{\frac{1}{8}}{\frac{9}{8}} = \frac{1}{9}.$

63. $r = \frac{4}{3} > 1$. So, no sum exists.

64. Base Case: Prove it is true for $n = 1$.

$2^1 = 2^{1+1} - 2 = 2$.

Thus, it is true for the base case.

Assume it is true for a natural number k.

Thus, $2 + 4 + \dots + 2^k = 2^{k+1} - 2$.

Look at $n = k + 1$.

$2 + 4 + \dots + 2^k + 2^{k+1}$

$= 2^{k+1} - 2 + 2^{k+1}$ (By Step 2)

$= 2(2^{k+1}) - 2$

$= 2^{k+2} - 2$

$= 2^{(k+1)+1} - 2$.

Thus, by induction, $2 + \dots + 2^n = 2^{n+1} - 2$

65. Base Case: Prove true for $n = 1$

$5^{1-1} = \frac{5^1 - 1}{4} = 1$.

Thus, it is true for the base case.

Assume it is true for a natural number k.

Thus, $1 + 5 + \dots + 5^{k-1} = \frac{5^k - 1}{4}$

Look at $n = k + 1$.

$1 + \dots + 5^{k-1} + 5^{k+1-1}$

$= \frac{5^k - 1}{4} + 5^k$

$= \frac{5^k - 1}{4} + \frac{4(5)^k}{4}$

$= \frac{5^k + 4(5)^k - 1}{4}$

$= \frac{(4 + 1)5^k - 1}{4}$

$= \frac{5(5)^k - 1}{4}$

$= \frac{5^{k+1} - 1}{4}$

Thus, by induction, $1 + 5 + \dots + 5^{n-1} = \frac{5^n - 1}{4}$

66. Base Case: Prove it is true for $n = 1$.

$\frac{1}{4(1^2) - 1} = \frac{1}{2(1) + 1} = \frac{1}{3}$

Thus, it is true for the base case.

Assume it is true for a natural number k.

Thus, $\frac{1}{3} + \dots + \frac{1}{4k^2 - 1} = \frac{k}{2k + 1}$

Look at $n = k + 1$.

$\frac{1}{3} + \dots + \frac{1}{4k^2 - 1} + \frac{1}{4(k + 1)^2 - 1}$

$= \frac{k}{2k + 1} + \frac{1}{4(k + 1)^2 - 1}$

$= \frac{k}{2k + 1} + \frac{1}{4k^2 + 8k + 3}$

$= \frac{k}{2k + 1} + \frac{1}{(2k + 1)(2k + 3)}$

$= \frac{k(2k + 3)}{(2k + 1)(2k + 3)} + \frac{1}{(2k + 1)(2k + 3)}$

$= \frac{2k^2 + 3k + 1}{(2k + 1)(2k + 3)}$

$= \frac{(2k + 1)(k + 1)}{(2k + 1)(2k + 3)}$

$= \frac{k + 1}{2k + 3}$

$= \frac{k + 1}{2(k + 1) + 1}$.

Thus, by induction $\frac{1}{3} + \dots + \frac{1}{4n^2 - 1} = \frac{n}{2n + 1}$

67a. $a_1 = 9$. $r = 0.85$. Thus, $S = \sum_{k=1}^{\infty} 9(0.85)^{k-1}$

b. $0.85 < 1$. Thus, this is an infinite converging geometric series. So:

$S = \frac{a_1}{1 - r} = \frac{9}{1 - 0.85} = \frac{9}{0.15} = 60$ ft.

CHAPTER TEST

1.

n	$n^2 - 4$	a_n
1	$(1)^2 - 4$	-3
2	$(2)^2 - 4$	0
3	$(3)^2 - 4$	5
4	$(4)^2 - 4$	12
5	$(5)^2 - 4$	21

2.

n	$\frac{1}{2}a_{n-1} - 8$	a_n
1	Given	48
2	$24 - 8$	16
3	$8 - 8$	0
4	$0 - 8$	-8